Power Series

A power series is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geqslant 1$

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a**. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$. Notice also that when $x = a$ all of the terms are 0 for $n \geqslant 1$ and so the power series (2) always converges when $x = a$.

EXAMPLE 1 For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

SOLUTION We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus, the given series converges only when $x = 0$.

EXAMPLE 2 For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION Let $a_n = (x-3)^n/n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\Sigma\, 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\Sigma\, (-1)^n/n$, which converges by the Alternating Series Test. Thus, the given power series converges for $2 \leqslant x < 4$.

3 **Theorem** For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

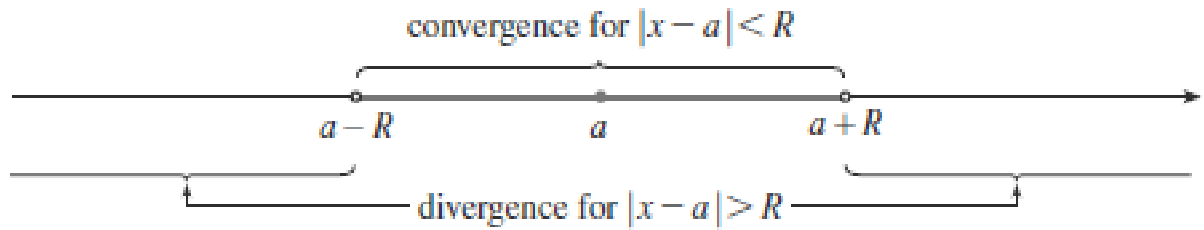
(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus, in case (iii) there are four possibilities for the interval of convergence:

$$(a-R, a+R) \qquad (a-R, a+R] \qquad [a-R, a+R) \qquad [a-R, a+R]$$



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We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n/\sqrt{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}\,|x| \to 3|x| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus, it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (Use the Integral Test or simply observe that it is a p-series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore, the given power series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$.

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus, the radius of convergence is $R = 3$.

The inequality $|x+2|<3$ can be written as $-5<x<1$, so we test the series at the endpoints -5 and 1. When $x=-5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x=1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus, the series converges only when $-5<x<1$, so the interval of convergence is $(-5, 1)$.

$\boxed{1}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

SOLUTION Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore, the interval of convergence is $(-1, 1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

Find a power series representation for $1/(x+2)$.

SOLUTION In order to put this function in the form of the left side of Equation 1 we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\dfrac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2, 2)$.

Find a power series representation of $x^3/(x+2)$.

SOLUTION Since this function is just x^3 times the function in Example 2, all we have to do is to multiply that series by x^3:

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$

$$= \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{8}x^5 - \tfrac{1}{16}x^6 + \cdots$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

the interval of convergence is $(-2, 2)$.

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration.**

$\boxed{2}$ **Theorem** If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$

(ii) $$\int f(x)\,dx = C + c_0(x - a) + c_1\frac{(x - a)^2}{2} + c_2\frac{(x - a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

NOTE 1 ▫ Equations (i) and (ii) in Theorem 2 can be rewritten in the form

$$\text{(iii)} \quad \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

$$\text{(iv)} \quad \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\, dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple;

NOTE 2 ▫ Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

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Express $1/(1-x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

we get
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

If we wish, we can replace n by $n+1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$.

Find a power series representation for $\ln(1 - x)$ and its radius of convergence.

SOLUTION We notice that, except for a factor of -1, the derivative of this function is $1/(1 - x)$. So we integrate both sides of Equation 1:

$$-\ln(1 - x) = \int \frac{1}{1 - x}\,dx = \int (1 + x + x^2 + \cdots)\,dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n + 1} + C$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} + C \qquad |x| < 1$$

To determine the value of C we put $x = 0$ in this equation and obtain $-\ln(1 - 0) = C$. Thus, $C = 0$ and

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$.

Notice what happens if we put $x = \frac{1}{2}$ in the result of Example . Since $\ln \frac{1}{2} = -\ln 2$, we see that

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

Evaluate $\int [1/(1+x^7)]\,dx$ as a power series.

(a) The first step is to express the integrand, $1/(1+x^7)$, as the sum of a power series.
we start with Equation 1 and replace x by $-x^7$:

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots$$

Now we integrate term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

Let's try to determine what the coefficients c_n must be in terms of f. To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

we can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

3 $$f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots \qquad |x - a| < R$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

4 $$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots \qquad |x - a| < R$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solving this equation for the nth coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus, we have proved the following theorem.

5 **Theorem** If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if* f *has a power series expansion at* a, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the Taylor series of the function f at a (or about a or centered at a). For the special case $a = 0$ the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name Maclaurin series.

NOTE ▫ We have shown that *if* f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series.

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

SOLUTION If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

The conclusion we can draw from Theorem 5 and Example   is that *if* e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x *does* have a power series representation?

Let's investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree n called the **nth-degree Taylor polynomial of f at a**. For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2$, and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \to \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

We have therefore proved the following.

8 **Theorem** If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following fact.

9 **Taylor's Inequality** If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

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$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x}$$

This is true because we know that the series $\Sigma\, x^n/n!$ converges for all x and so its nth term approaches 0.

EXAMPLE 2 Prove that e^x is equal to the sum of its Maclaurin series.

SOLUTION If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \leqslant d$, then $|f^{(n+1)}(x)| = e^x \leqslant e^d$. So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

Notice that the same constant $M = e^d$ works for every value of n. But, from Equation 10, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n\to\infty}|R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x. By Theorem 8, e^x is equal to the sum of its Maclaurin series, that is,

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$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x}$$

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

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$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots}$$

Find the Taylor series for $f(x) = e^x$ at $a = 2$.

SOLUTION We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Again it can be verified, that the radius of convergence is $R = \infty$. As in Example we can verify that $\lim_{n \to \infty} R_n(x) = 0$, so

13 $$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x$$

Where does it converge?

Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

SOLUTION We arrange our computation in two columns as follows:

$$\begin{aligned} f(x) &= \sin x & \qquad f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we know that $|f^{(n+1)}(x)| \leqslant 1$ for all x. So we can take $M = 1$ in Taylor's Inequality:

14 $$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 10 the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series by Theorem 8.

15 $$\boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x \end{aligned}}$$

We have denoted the sum by $g(x)$ since we don't yet know whether the series converges to $\sin x$. The series does converge for all x by the ratio test:

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}}{(2(n+1)+1)!}x^{2(n+1)+1}}{\dfrac{(-1)^n}{(2n+1)!}x^{2n+1}}\right| = \lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}|x|^2$$

$$= \lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)} = 0.$$

Find the Maclaurin series for $\cos x$.

SOLUTION We could proceed directly as in Example 4 but it's easier to differentiate the Maclaurin series for $\sin x$ given by Equation 15:

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Since the Maclaurin series for $\sin x$ converges for all x, Theorem tells us that the differentiated series for $\cos x$ also converges for all x. Thus

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$$\boxed{\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x\end{aligned}}$$

Other Maclaurin and Taylor Series

Series can be combined in various ways to generate new series. For example, we can find the Maclaurin series for e^{-x} by replacing x with $-x$ in the series for e^x:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \quad \text{(for all } x\text{)}.$$

The series for e^x and e^{-x} can then be subtracted or added and the results divided by 2 to obtain Maclaurin series for the hyperbolic functions $\sinh x$ and $\cosh x$:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{(for all } x\text{)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{(for all } x\text{)}.$$

Obtain Maclaurin series for the following functions:

(a) $e^{-x^2/3}$, (b) $\dfrac{\sin(x^2)}{x}$, (c) $\sin^2 x$.

Solution

(a) We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

$$e^{-x^2/3} = 1 - \frac{x^2}{3} + \frac{1}{2!}\left(\frac{x^2}{3}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{3}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n n!} x^{2n} \qquad \text{(for all real } x\text{)}.$$

(b) For all $x \neq 0$ we have

$$\frac{\sin(x^2)}{x} = \frac{1}{x}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right)$$

$$= x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n+1)!}.$$

Note that $f(x) = (\sin(x^2))/x$ is not defined at $x = 0$ but does have a limit (namely 0) as x approaches 0. If we define $f(0) = 0$ (the continuous extension of $f(x)$ to $x = 0$), then the series converges to $f(x)$ for all x.

(c) We use a trigonometric identity to express $\sin^2 x$ in terms of $\cos 2x$ and then use the Maclaurin series for $\cos x$ with x replaced by $2x$.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)$$

$$= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+2)!} x^{2n+2} \qquad \text{(for all real } x\text{)}.$$

Find the Taylor series for $\ln x$ in powers of $x - 2$. Where does the series converge to $\ln x$?

Solution Note that if $t = (x-2)/2$, then

$$\ln x = \ln\big(2 + (x-2)\big) = \ln\left[2\left(1 + \frac{x-2}{2}\right)\right] = \ln 2 + \ln(1+t).$$

We use the known Maclaurin series for $\ln(1+t)$:

$$\begin{aligned}
\ln x &= \ln 2 + \ln(1+t) \\
&= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} - \cdots \\
&= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2 \times 2^2} + \frac{(x-2)^3}{3 \times 2^3} - \frac{(x-2)^4}{4 \times 2^4} + \cdots \\
&= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 2^n} (x-2)^n.
\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < (x-2)/2 \le 1$, that is, for $0 < x \le 4$.

Find the Maclaurin series for the function $f(x) = x \cos x$.

SOLUTION Instead of computing derivatives and substituting in Equation 7, it's easier to multiply the series for $\cos x$ (Equation 16) by x:

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

SOLUTION Arranging our work in columns, we have

$$f(x) = \sin x \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

and this pattern repeats indefinitely. Therefore, the Taylor series at $\pi/3$ is

$$f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

The proof that this series represents $\sin x$ for all x is very similar to that in Example 4. [Just replace x by $x - \pi/3$ in (14).] We can write the series in sigma notation if we separate the terms that contain $\sqrt{3}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1, 1]$$

Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(a) First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it simply by replacing x with $-x^2$ in the series for e^x given in the table of Maclaurin series. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Now we integrate term by term:

$$\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$

$$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

$$\text{Evaluate } \lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$

SOLUTION Using the Maclaurin series for e^x, we have

$$\begin{aligned}
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \to 0} \frac{\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right) - 1 - x}{x^2} \\
&= \lim_{x \to 0} \frac{\dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots}{x^2} \\
&= \lim_{x \to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right) \\
&= \frac{1}{2}
\end{aligned}$$

because power series are continuous functions.

Obtain the first three nonzero terms of the Maclaurin series for (a) $\tan x$ and (b) $\ln \cos x$.

Solution

(a) $\tan x = (\sin x)/(\cos x)$. We can obtain the first three terms of the Maclaurin series for $\tan x$ by long division of the series for $\cos x$ into that for $\sin x$:

(b) $$\ln\cos x = \ln\left(1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right)$$

$$= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2$$

$$+ \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^3 - \cdots$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots - \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right)$$

$$+ \frac{1}{3}\left(-\frac{x^6}{8} + \cdots\right) - \cdots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots.$$

Observe that the series for $\tan x$ can also have been derived from that of $\ln\cos x$ because we have $\tan x = -\frac{d}{dx}\ln\cos x$.

EXAMPLE 8 Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

SOLUTION Arranging our work in columns, we have

$$\begin{aligned}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1 + x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**. If its nth term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \to |x| \quad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. □

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1+x)^k$ is equal to the sum of its Maclaurin series.

THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. It turns out that the series converges at 1 if $-1 < k \leqslant 0$ and at both endpoints if $k \geqslant 0$.

Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\dfrac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3\right.$$

$$\left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!\,8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!\,8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,8^n}x^n + \cdots\right]$$

We know that this series converges when $|-x/4| < 1$, that is, $|x| < 4$, so the radius of convergence is $R = 4$.

Find the Maclaurin series for $\dfrac{1}{\sqrt{1+x}}$.

Solution Here $r = -(1/2)$:

$$\begin{aligned}
\frac{1}{\sqrt{1+x}} &= (1+x)^{-1/2} \\
&= 1 - \frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3 + \cdots \\
&= 1 - \frac{1}{2}x + \frac{1\times 3}{2^2 2!}x^2 - \frac{1\times 3\times 5}{2^3 3!}x^3 + \cdots \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}x^n.
\end{aligned}$$

This series converges for $-1 < x \leq 1$. (Use the alternating series test to get the endpoint $x = 1$.)

Find the Maclaurin series for $\sin^{-1} x$.

Solution Replace x with $-t^2$ in the series obtained in the previous example to get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!} t^{2n} \qquad (-1 < t < 1).$$

Now integrate t from 0 to x:

$$\begin{aligned}
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!} t^{2n}\right) dt \\
&= x + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!(2n+1)} x^{2n+1} \\
&= x + \frac{x^3}{6} + \frac{3}{40} x^5 + \cdots \qquad (-1 < x < 1).
\end{aligned}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Evaluate (a) $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$ and (b) $\lim_{x \to 0} \dfrac{(e^{2x} - 1)\ln(1 + x^3)}{(1 - \cos 3x)^2}$.

Solution

(a) $\lim_{x \to 0} \dfrac{x - \sin x}{x^3} \qquad \left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{\dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \cdots}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right) = \frac{1}{3!} = \frac{1}{6}.$$

(b) $\displaystyle\lim_{x\to 0}\frac{(e^{2x}-1)\ln(1+x^3)}{(1-\cos 3x)^2} \qquad \left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\left(1+(2x)+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots-1\right)\left(x^3-\frac{x^6}{2}+\cdots\right)}{\left(1-\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\cdots\right)\right)^2}$$

$$= \lim_{x\to 0}\frac{2x^4+2x^5+\cdots}{\left(\frac{9}{2}x^2-\frac{3^4}{4!}x^4+\cdots\right)^2}$$

$$= \lim_{x\to 0}\frac{2+2x+\cdots}{\left(\frac{9}{2}-\frac{3^4}{4!}x^2+\cdots\right)^2} = \frac{2}{\left(\frac{9}{2}\right)^2} = \frac{8}{81}.$$

You can check that the second of these examples is much more difficult if attempted using l'Hôpital's Rule.

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

SOLUTION

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rl}
 & 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots \\
\times & x \qquad\qquad - \tfrac{1}{6}x^3 + \cdots \\
\hline
 & x + \quad x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \\
+ & \qquad\qquad - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 - \cdots \\
\hline
 & x + \quad x^2 + \tfrac{1}{3}x^3 + \cdots
\end{array}$$

Thus

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

We use a procedure like long division:

$$\begin{array}{r l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\cline{2-2}
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \Big) & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \qquad \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
 & \qquad \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \qquad\qquad \frac{2}{15}x^5 + \cdots
\end{array}$$

Thus
$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

Although we have not attempted to justify the formal manipulations used in Example they are legitimate. There is a theorem which states that if both $f(x) = \Sigma\, c_n x^n$ and $g(x) = \Sigma\, b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

$$\int_1^{\infty} \frac{1+\cos\sqrt{x}}{1+\sqrt{x}+x^4}\,dx \qquad -1 \leq \cos\sqrt{x} \leq 1 \;;\; x \in [1,\infty)$$

$$0 \leq 1+\cos\sqrt{x} \leq 2$$

$$0 \leq \frac{1+\cos\sqrt{x}}{1+\sqrt{x}+x^4} < \frac{2}{1+\sqrt{x}+x^4} < \frac{2}{x^4}$$

$$\int_1^{\infty} \frac{2}{x^4}\,dx = 2\int_1^{\infty} \frac{1}{x^4}\,dx \qquad p=4>1$$

İs convergent(p-integral,p=4>1)

Thus, $\displaystyle\int_1^{\infty} \frac{1+\cos\sqrt{x}}{1+\sqrt{x}+x^4}\,dx$

İs convergent

OR

$$\int_1^\infty \frac{1}{x^4}dx = \lim_{b\to\infty} -\frac{1}{3x^3}\Big|_1^b = \lim_{b\to\infty} \underbrace{-\frac{1}{3b^3}}_{0} + \frac{1}{3} = \frac{1}{3}$$ yakınsaktır,

$$\left(2\int_1^\infty \frac{1}{x^4}dx = \frac{2}{3} \text{ olur} \right)$$

$$\int_1^{\infty}\frac{2}{x\sqrt{x^2-1}}\,dx$$

Çözüm:

Such that

$c>1$

$$\int_1^{\infty}\frac{2}{x\sqrt{x^2-1}}\,dx=\underbrace{\int_1^{c}\frac{2}{x\sqrt{x^2-1}}\,dx}+\underbrace{\int_c^{\infty}\frac{2}{x\sqrt{x^2-1}}\,dx}$$

$$\int\frac{2}{x\sqrt{x^2-1}}\,dx=\int\frac{2\tan\theta\sec\theta\,d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}=2\,\mathrm{arcsec}\,x+C$$

$$x=\sec\theta\Rightarrow dx=\tan\theta\sec\theta\,d\theta$$

Let

$c=2$

$$\underbrace{\int_1^{2}\frac{2}{x\sqrt{x^2-1}}\,dx}_{\text{II. Tip improper}}=\lim_{a\to1^+}\int_a^{2}\frac{2}{x\sqrt{x^2-1}}\,dx=2\lim_{a\to1^+}\left(\mathrm{arcsec}\,x\Big|_a^2\right)=2\lim_{a\to1^+}\left(\frac{\pi}{3}-\text{arcsec a}\right)=\frac{2\pi}{3}$$

$$\underbrace{\int_2^{\infty}\frac{2}{x\sqrt{x^2-1}}\,dx}_{\text{I. Tip improper}}=\lim_{b\to\infty}\int_2^{b}\frac{2}{x\sqrt{x^2-1}}\,dx=2\lim_{b\to\infty}\left(\mathrm{arcsec}\,x\Big|_2^b\right)=2\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\frac{\pi}{3}$$

$$\boxed{\int_1^{\infty}\frac{2}{x\sqrt{x^2-1}}\,dx=\frac{2\pi}{3}+\frac{\pi}{3}=\pi}$$

a) $I = \int_0^2 x^2 \ln x\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^2 x^2 \ln x\, dx$

$\ln x = u \qquad x^2 dx = dv$

$\frac{1}{x} dx = du \qquad \frac{x^3}{3} = v$

$$I = \lim_{\varepsilon \to 0} \left[\frac{1}{3} x^3 \cdot \ln x \Big|_\varepsilon^2 - \frac{1}{3} \int_\varepsilon^2 x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}} dx \right]$$

$$I = \lim_{\varepsilon \to 0} \left[\frac{1}{3} (8 \ln 2 - \varepsilon^3 \ln \varepsilon) - \frac{1}{3} \cdot \frac{x^3}{3} \Big|_\varepsilon^2 \right]$$

$$I = \lim_{\varepsilon \to 0} \left[\frac{8}{3} \ln 2 - \frac{1}{3} \underbrace{\varepsilon^3 \ln \varepsilon}_{A} - \frac{8}{9} + \frac{\varepsilon^3}{9} \right]$$

$$A = \lim_{\varepsilon \to 0} \varepsilon^3 \cdot \ln \varepsilon = 0 \cdot (-\infty)$$

$$A = \lim_{\varepsilon \to 0} \frac{\ln \varepsilon}{\frac{1}{\varepsilon^3}} = \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}}{-\frac{3}{\varepsilon^4}} = \lim_{\varepsilon \to 0} -\frac{1}{3}\varepsilon^3 = 0$$

$$\boxed{I = \frac{8}{3}\ln 2 - \frac{8}{9}}$$

1) Determine whether the following integrol is convergent or divergent

$$\int_1^{\infty} \frac{\cos^2 x}{x^2+1}\, dx$$

2) Determine whether the following series are convergent or divergent

a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

b) $\sum_{n=3}^{\infty} \frac{5n^4-3n^3+2}{n^2\cdot(n+1)(n^2+5)}$

$$\int_1^\infty \frac{\cos^2 x}{x^2+1}\,dx \qquad \text{(Direct comparison test)}$$

$$\frac{\cos^2 x}{x^2+1} \leq \frac{1}{x^2+1} < \frac{1}{x^2}.$$

For $\forall x \in [1,\infty)$, $\int_1^\infty \frac{1}{x^2}\,dx$ (p-integral. $p=2>1$),

so, it is <u>convergent</u>

Thus, $\int_1^\infty \frac{\cos^2 x}{x^2+1}\,dx$ is also <u>convergent</u>.

1 a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Let $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$ for $n \geq 1$. Then $a_n > 0$ for all n and

$\sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^{-n}$. Hence,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = e^{-1} < 1$$

By root test, we find that the series is convergent.

b) $\sum_{n=3}^{\infty} \frac{5n^4-3n^3+2}{n^2\cdot(n+1)(n^2+5)}$

Let $a_n = \frac{5n^4-3n^3+2}{n^2\cdot(n+1)(n^2+5)}$, $b_n = \frac{1}{n}$ Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^4-3n^3+2}{n^2(n+1)(n^2+5)} \cdot \frac{n}{1} = 5 > 0$$

By p-test, we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Hence, by limit comparison test, the given series is divergent.

Determine whether the following integrol convergent or divergent

$$\int_1^\infty \frac{1+\sin^2(e^{\cos x})}{\sqrt{x}}\,dx .$$

a) Determine if the following series is convergent. If so, find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

b) $$\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx = ?$$

For $\forall x \geqslant 1 \quad 1+\sin^2(e^{\cos x}) \geqslant 1$

Because, For $\forall x \geqslant 1 \quad \sin^2(e^{\cos x}) = \left(\sin(e^{\cos x})\right)^2 \geqslant 0$

For $\forall x \geqslant 1 \quad 0 \leqslant \dfrac{1}{\sqrt{x}} \leqslant \dfrac{1+\sin^2(e^{\cos x})}{\sqrt{x}}$

$\displaystyle\int_1^\infty \frac{1+\sin^2(e^{\cos x})}{\sqrt{x}}\,dx$ is <u>divergent</u> <u>because</u>

$\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ <u>divergent</u>. (Direct comparison test)

a) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

The portial sums of this series

$$S_K = \sum_{n=1}^{K} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{K} \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cdots + \frac{1}{2K-1} - \frac{1}{2K+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2K+1}\right)$$

$\lim_{K \to \infty} S_K = \frac{1}{2}$ and the series converges.

The sum of the series is the limit of the portial sums.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

$$\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx = ?$$

$$\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}} = \lim_{b\to\frac{\pi}{6}} \int_0^{b} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx \qquad \begin{aligned} 1-2\sin x &= t \\ -2\cos x\,dx &= dt \end{aligned}$$

$$\lim_{b\to\frac{\pi}{6}} -\frac{1}{2}\int_0^{b} \frac{dt}{\sqrt{t}} = \lim_{b\to\frac{\pi}{6}} -\frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{t}\,\Big|_0^{b} = \lim_{\varepsilon\to 0} -\sqrt{1-2\sin x}\,\Big|_0^{b}$$

$$= \lim_{b\to 0} -\left[\sqrt{1-2\sin\left(\tfrac{\pi}{6}\right)} - 1\right] = 1$$

$$\sum_{n=1}^{\infty} \left[\arctan(n+1) - \arctan(n)\right]$$ is given.

Find the sum of the given series

$$\sum_{a=3}^{\infty} \frac{1}{a \cdot \ln a \sqrt{\ln(\ln a)}}$$ is given.

Determine if it is convergent or divergent

$$\sum_{n=1}^{\infty}\left[\arctan(n+1)-\arctan(n)\right]$$

$$S_k=\sum_{n=1}^{k}\left[\arctan(n+1)-\arctan(n)\right]$$

$$=\cancel{\arctan 2}-\underline{\arctan 1}+\arctan 3-\cancel{\arctan 2}+\cdots+\underline{\arctan(k+1)}-\arctan k\,]$$

$$\lim_{k\to\infty}S_k=\lim_{k\to\infty}\left[\arctan(k+1)-\arctan 1\right]$$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$$\sum_{a=3}^{\infty} \frac{1}{a \cdot \ln a \cdot \sqrt{\ln(\ln a)}}$$

For $\forall n$; $u_{n+1} < u_n$. So, Apply the integral test

$$\int_3^{\infty} \frac{da}{a \ln a \sqrt{\ln(\ln a)}} = \lim_{b\to\infty} \int_3^b \frac{du}{u \cdot \sqrt{\ln u}}$$

$\ln u = t$

$\frac{1}{u} du = dt$

$\ln a = u$

$\frac{1}{a} da = du$

$$= \lim_{b\to\infty} \int_3^b \frac{dt}{\sqrt{t}}$$

$$= \lim_{b\to\infty} \left(2\sqrt{t}\right)\Big|_3^b$$

$$= \lim_{b\to\infty} \left(2\sqrt{\ln(\ln a}\right)\Big|_3^b \longrightarrow \infty$$

Divergent.

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$ is given.

Determine if it is convergent or divergent

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) \longrightarrow$$ (Apply integrol test)

<u>It will be divergent</u>

$$\int_1^\infty \frac{(\ln x)\cdot \sin^2 x}{x^3+2}\,dx$$

Since $0 \le \sin^2 x \le 1$ we conclude that

$$\int_1^\infty \frac{(\ln x)\cdot \sin^2 x}{x^3+2}\,dx \le \int_1^\infty \frac{\ln x}{x^3+2}\,dx$$

For x in $[1,\infty)$ we know that $\ln x < x$ and so

$$\int_1^\infty \frac{\ln x}{x^3+2}\,dx < \int_1^\infty \frac{x}{x^3+2}\,dx$$

Since for a rational function the highest powers of x dominate as x goes to ∞ we have $\frac{x}{x^3+2} \sim \frac{1}{x^2}$ as $x \to \infty$

By the p-test we know that $\int_1^\infty \frac{dx}{x^2}$ converges, so by the limit comparison test we know that $\int_1^\infty \frac{x}{x^3+2}\,dx$ also converges, and by the comparison test we conclude that the original integral also converges.

b) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \cdot \frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| \cdot |x| = \lim_{n\to\infty}\left|\frac{1}{(1+\frac{1}{n})^n}\right| |x|$$

$$= \frac{1}{e}|x| < 1 \Rightarrow |x| < e$$

The interval of convergence $-e < x < e$

$R = e$

For x=e the given series diverges.
For x=-e the given series diverges.

Please show it .