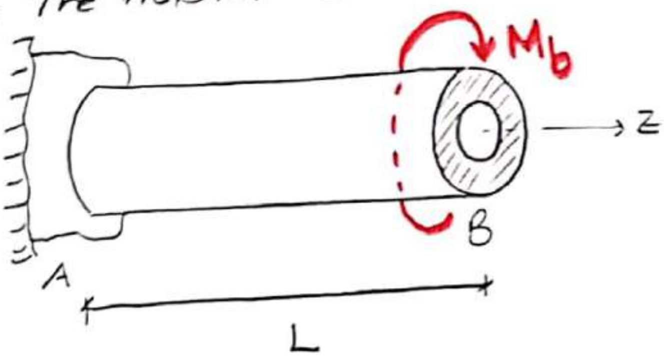
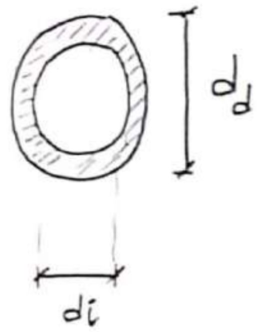
Example: The hollow shaft is subjected to the torque of $M_b = 17{,}1$ kNm. This shaft is fixed to the support at A. Determine the inner diameter d_i of the shaft to the nearest mm if the maximum outer diameter of the shaft can be 100 mm. Sketch the shear stress distribution on the cross-section. The material has an allowable shear stress of $\tau_{allow} = 120$ MPa





Cross-section

$$\tau_{max} \leq \tau_{allow}$$

$$\tau = \frac{M_b}{I_o} \cdot r \quad , \text{ where } I_o = \frac{\pi}{32}\left(d_d^4 - d_i^4\right) \text{ and } r_d \geq r \geq r_i$$

when $r = 1/2 \, d_d$; $\tau = \tau_{max}$

$$\frac{M_b}{I_o}\left(\frac{1}{2} d_d\right) \leq \tau_{allow}$$

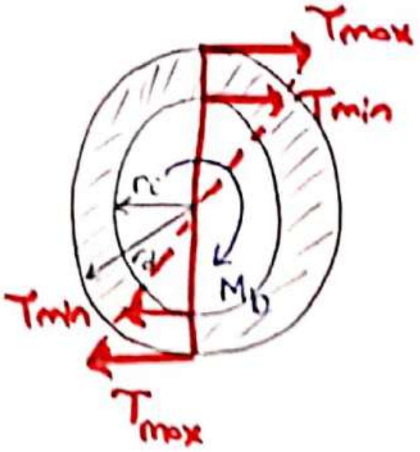
$$\frac{M_b}{\frac{\pi}{32}\left(d_d^4 - d_i^4\right)}\left(d_d/2\right) \leq \tau_{allow}$$

$$d_i^4 \leq d_d^4 \left[1 - \frac{16 \, M_b}{\tau_{allow} \cdot \pi \cdot d_d^3}\right]$$

$$d_i \leq d_d \sqrt[4]{1 - \frac{16 \, M_b}{\tau_{allow} \cdot \pi \, d_d^3}}$$

$$d_i \leq 100 \sqrt[4]{1 - \frac{16\,(17{,}1 \times 10^6)}{\pi \cdot 100^3 \cdot 120}} \;\; \tilde{\leq} \;\; \underline{\underline{72{,}36 \text{ mm}}}$$

So, $d_i = 72$ mm (con bachoseras 72 mm)



$$\tau_{min} = \frac{16 M_b d_i}{\pi d_d^4 \left[1 - (d_i/d_d)^4\right]} = 85{,}75 \text{ MPa}$$

$$\tau_{min} = \frac{M_b}{I_0} \cdot r$$

$$\tau_{min} = \frac{M_b}{\frac{\pi}{32}\left(d_d^4 - d_i^4\right)} \left(\frac{d_i}{2}\right)$$

$$\tau_{min} = \frac{16 M_b d_i}{\pi d_d^4 \left[1 - \frac{d_i^4}{d_d^4}\right]} = \frac{16(17{,}1 \cdot 10^6)(72)}{\pi 100^4 \left[1 - \frac{72^4}{100^4}\right]}$$

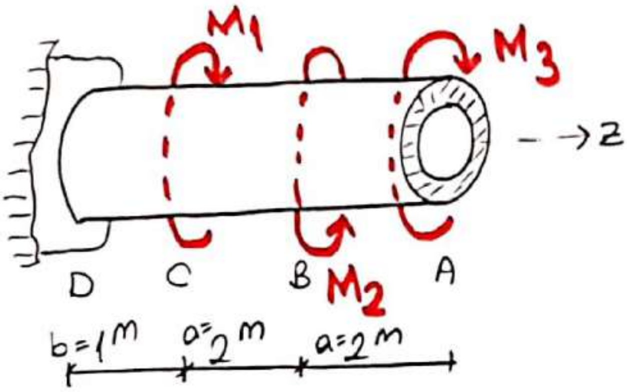
$$\tau_{min} = 85.75 \text{ MPa}$$

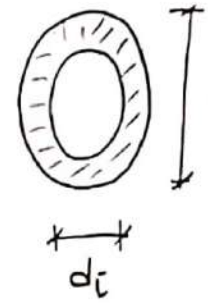
Example: The hollow shaft is fixed to the support at end D and free at end A subjected to the torsional loadings. The shaft has an inner diameter of 50 mm and outer diameter of 100 mm. If the allowable shear stress for the shaft's material is $\tau_{allow} = 80$ MPa, $E = 200$ GPa and $\nu = 0,25$.

a) determine the maximum allowable torque M_0

b) draw the torsional moment diagram

c) what's the angle of twist A relative to D ($\varphi_{A/D}$)

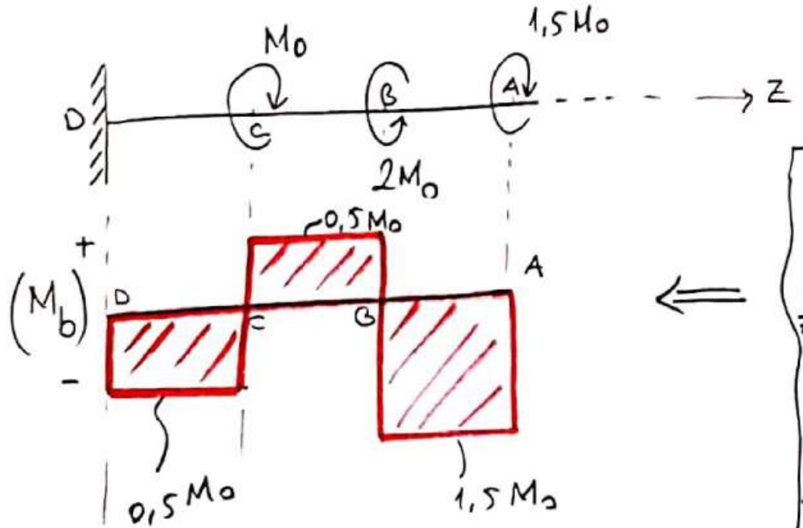




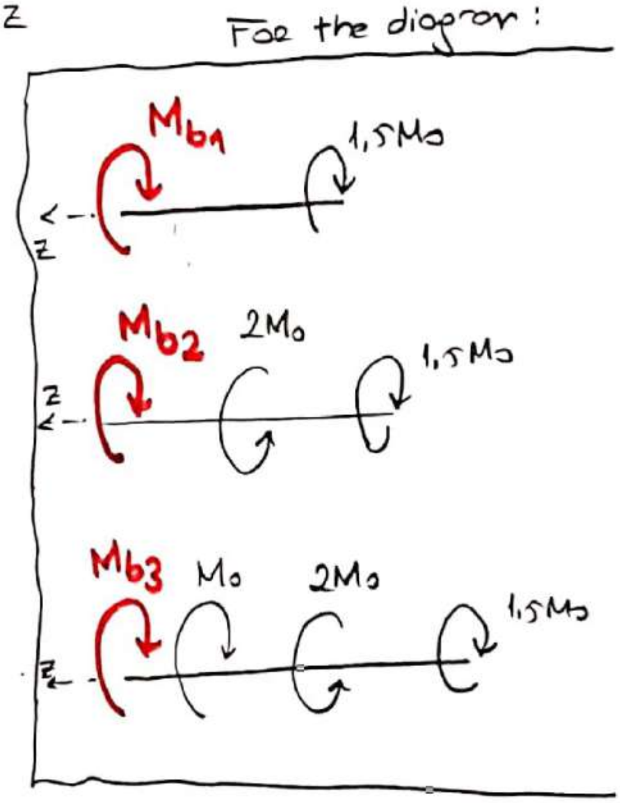
$M_1 = M_0$

$M_2 = 2M_0$

$M_3 = 1,5 M_0$



For the diagram:



$$(M_b)_{max} = |-1,5 M_0|$$

$$\tau_{max} = \frac{(M_b)_{max}}{I_0} \cdot r_d \leq \tau_{allow}$$

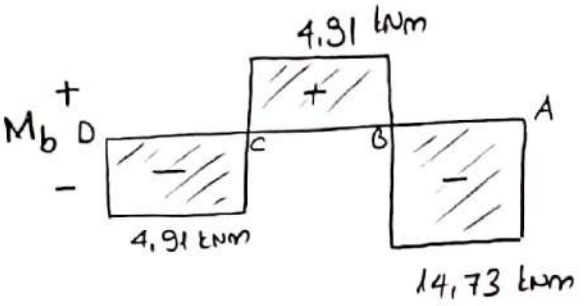
$$\left.\begin{aligned} r_d &= 1/2\, d_d = 50 \text{ mm} \\ r_i &= 25 \text{ mm} \end{aligned}\right\} \quad I_0 = \frac{1}{2}\pi\left(r_d^4 - r_i^4\right) = 9,2 \times 10^6 \text{ mm}^4$$

$$M_0 \le \frac{\tau_{allow} \cdot I_o}{1{,}5\, r_d} = \frac{80\,(9{,}2 \cdot 10^6)}{1{,}5\,(50)} \simeq 9.82 \times 10^6 \text{ Nmm}$$

$$\simeq 9.82 \text{ kNm}$$

$M_1 \cong 9.82$ kNm ; $M_2 \cong 19.64$ kNm ; $M_3 \cong 14.73$ kNm

b)



c) $$\phi_{AD} = \phi_{CD} + \phi_{BC} + \phi_{AB} = \frac{(M_b)_{CD} \cdot b}{G I_o} + \frac{(M_b)_{BC} \cdot a}{G I_o} + \frac{(M_b)_{AB} \cdot a}{G I_o}$$

$$G = E/2(1+\nu) = 200/2(1+0{,}25) = 80 \text{ GPa}$$

$$G I_o = 736 \text{ kNm}^2$$

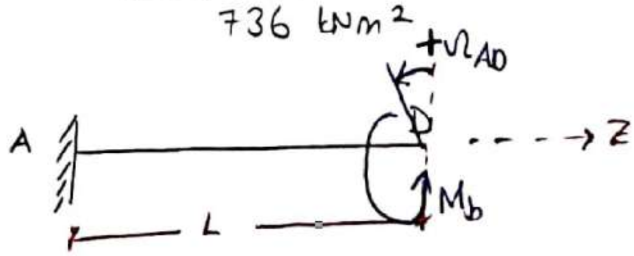
$(M_b)_{AB} = -14{,}73 \text{ kNm} = -1{,}5 M_o$

$(M_b)_{BC} = 0{,}5 M_o = 4.91 \text{ kNm}$

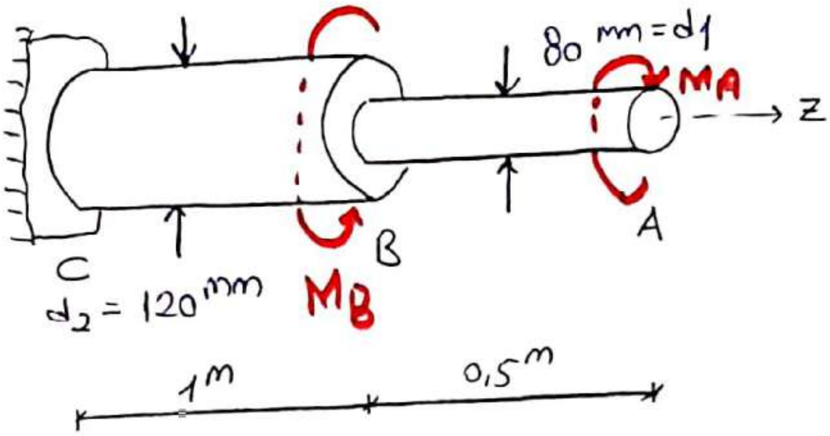
$(M_b)_{CD} = -0{,}5 M_o = -4.91 \text{ kNm}$

$$\phi_{AD} = \frac{(-0{,}5 M_o) \cdot 1}{G I_o} + \frac{(0{,}5 M_o) \cdot 2}{G I_o} + \frac{(-1{,}5 M_o) \cdot 2}{G I_o}$$

$$\phi_{AD} = \frac{-2{,}5 * 9.82 \text{ kNm}}{736 \text{ kNm}^2} = -0{,}033 \text{ rad.}$$



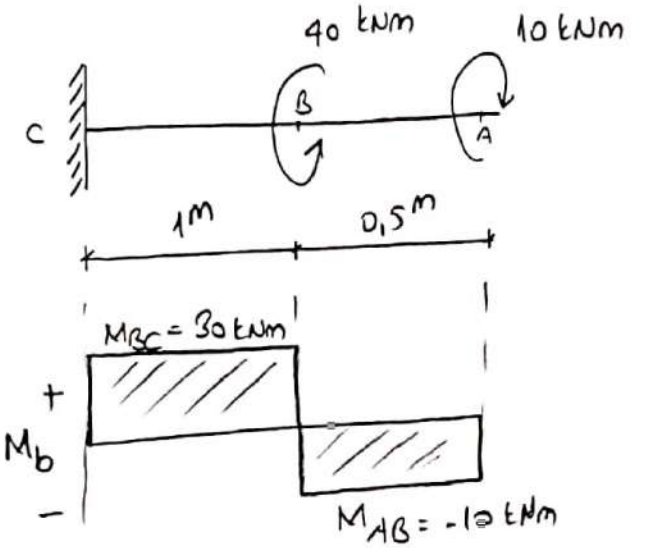
Example: The variable diameter shaft is subjected to the torques shown. Determine a) the maximum shear stress developed in the shaft b) the angle of twist of B with respect to C c) the angle of twist of A relative to C.

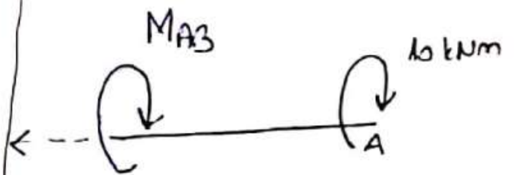


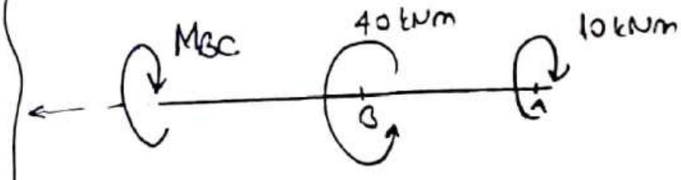
$G = 80\ GPa$

$M_B = 40\ kNm$

$M_A = 10\ kNm$







$$(I_o)_{AB} = \frac{1}{2}\pi . 40^4 = \frac{1}{2}\pi r^4 \cong 402 \times 10^4\ mm^4$$

$$(I_o)_{BC} = \frac{1}{2}\pi\, 60^4 = \frac{1}{2}\pi r^4 \cong 2036 . 10^4\ mm^4$$

$$(\tau_{max})_{AB} = \frac{M_{AB} . r_{AB}}{(I_o)_{AB}} = \frac{|-10.10^6|}{402.10^4} . 40 = 99,5\ MPa$$

$$(\tau_{max})_{BC} = \frac{M_{BC} . r_{BC}}{(I_o)_{BC}} = \frac{30.10^6}{2036.10^4} . 60 = 88,4\ MPa$$

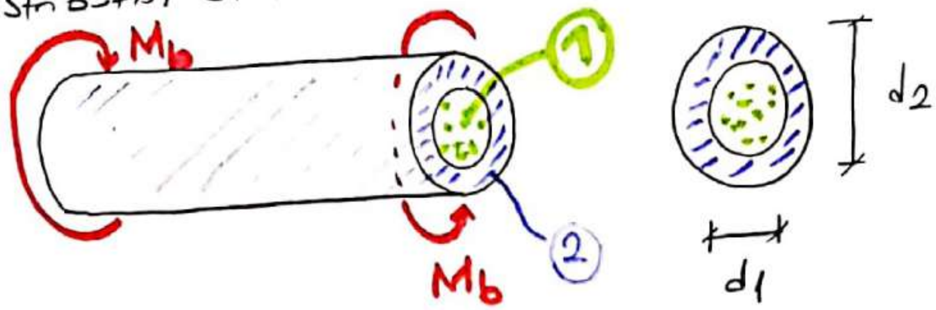
b) $\varphi_{BC} = \dfrac{M_{BC} \cdot L_{BC}}{G(I_0)_{BC}} = \dfrac{(30 \cdot 10^6)(10^3)}{(80 \cdot 10^3)(2036 \cdot 10^4)} = 0,0184 \text{ rad.}$

c) $\varphi_{AC} = \varphi_{BC} + \varphi_{AB} = 0,0184 + \underbrace{\dfrac{M_{AB} \cdot L_{AB}}{G(I_0)_{AB}}} = 0,00287 \text{ rad}$

$\Downarrow$

$$\frac{(-10 \cdot 10^6)(500)}{(80 \cdot 10^3)(402 \cdot 10^4)}$$

Example: The shaft shown in Figure is made from material ①, which is bonded to material ②. The diameter of the core material (material ①) is d_1 and the coating material (material ②) is d_2. The relationship between the shear modulus of the materials is $G_2 > G_1$. Determine the angle of twist and sketch the shear stress distribution on the cross-section.



$$M_b = M_1 + M_2 \quad \text{--- (A)}$$

$$w_1 = \frac{M_1}{G_1 (I_0)_1} \quad ; \quad w_2 = \frac{M_2}{G_2 (I_0)_2} \quad \Rightarrow \quad w_1 = w_2 = w \quad \text{--- (B)}$$

$$\left.\begin{aligned} w_1 &= \frac{32 M_1}{G_1 \pi d_1^4} \\ w_2 &= \frac{32 M_2}{G_2 \pi (d_2^4 - d_1^4)} \end{aligned}\right\} \quad \text{--- (C)}$$

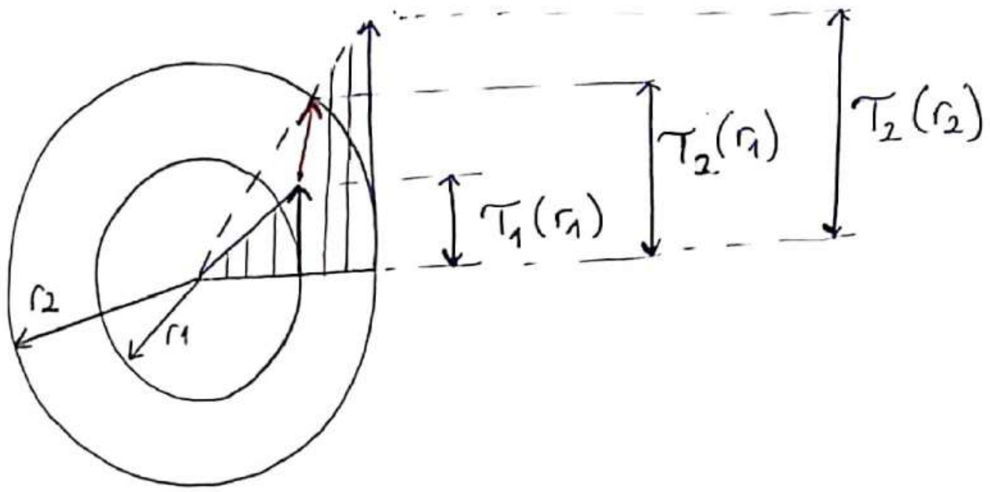
$$M_2 = \underbrace{\frac{G_2}{G_1}}_{\alpha} M_1 \frac{(d_2^4 - d_1^4)}{d_1^4} \quad \text{--- (D)} \quad \text{(Considering Eqs (C) and (B))}$$

Let's substitute Eq (D) into Eq (A)

$$\left.\begin{aligned} M_1 &= \frac{d_1^4}{d_1^4 + \alpha (d_2^4 - d_1^4)} M_b \\ M_2 &= \frac{\alpha (d_2^4 - d_1^4)}{d_1^4 + \alpha (d_2^4 - d_1^4)} M_b \end{aligned}\right\} \quad \text{--- (E)}$$

From Eqs (C) and (E) ;

$$\omega_1 = \omega_2 = \frac{32}{G\pi\left[d_1^4 + \alpha\left(d_2^4 - d_1^4\right)\right]} M_b$$



Shear stress distribution

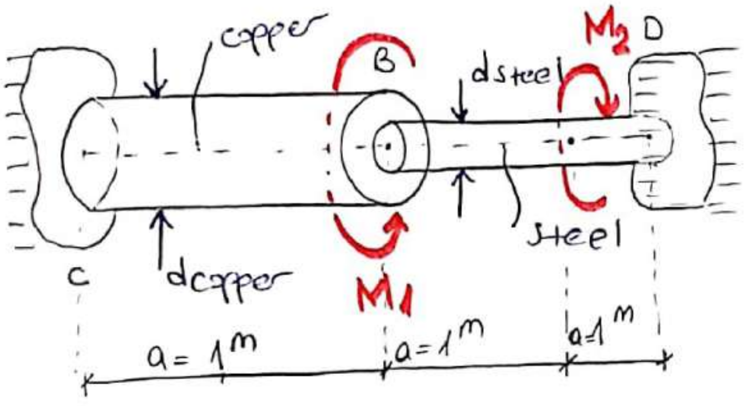
$$\tau_1\Big|_{r=\frac{1}{2}d_1} = \frac{M_1\left(\frac{1}{2}d_1\right)}{(I_o)_1} = \frac{16\, d_1 M_b}{\pi\left[d_1^4 + \alpha\left(d_2^4 - d_1^4\right)\right]}$$

$$\tau_2\Big|_{r=\frac{1}{2}d_1} = \frac{M_2\left(\frac{1}{2}d_1\right)}{(I_o)_2} = \frac{16\,\alpha\, d_1 M_b}{\pi\left[d_1^4 + \alpha\left(d_2^4 - d_1^4\right)\right]}$$

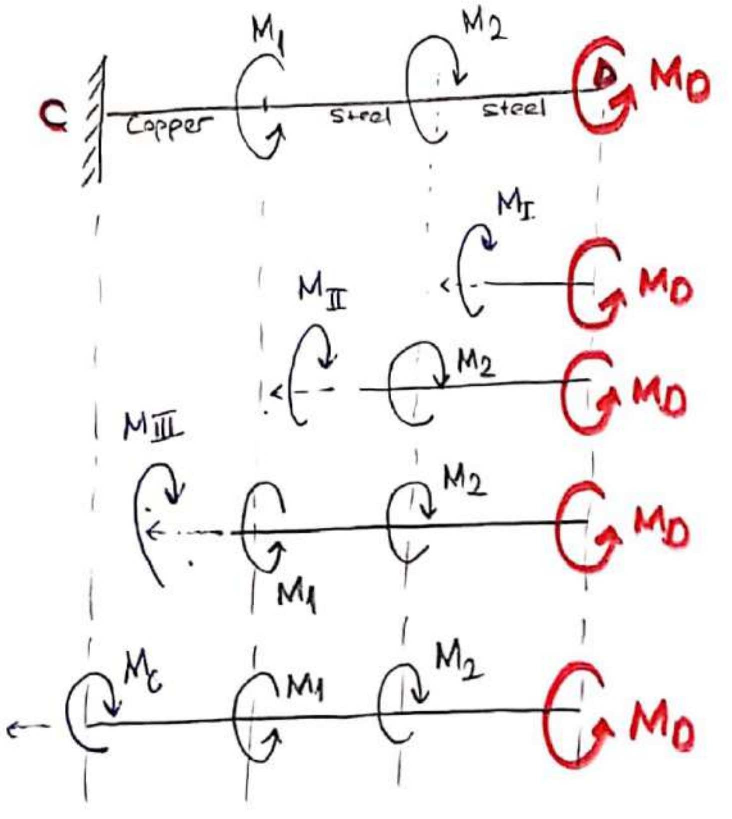
$$\tau_{max} = \tau_2\Big|_{r=\frac{1}{2}d_2} = \frac{M_2\left(\frac{1}{2}d_2\right)}{(I_o)_2} = \frac{16\,\alpha\, d_2 M_b}{\pi\left[d_1^4 + \alpha\left(d_2^4 - d_1^4\right)\right]}$$

Example: The solid shaft is made from 2 segments: steel section BD and copper section CB. The allowable shear stresses for the steel and copper materials are 80 MPa and 50 MPa, respectively. And $G_{copper} = 30$ GPa, $G_{steel} = 80$ GPa

a) Draw the torsional moment diagram
b) Determine the required diameters of the steel and copper sections of the shaft.



$M_1 = 40$ kNm
$M_2 = 20$ kNm
$d_{copper} = 1{,}2\, d_{steel}$



$M_I = M_D \quad (0 \le z \le 1^m)$

$M_{II} = M_D - 20 \quad (1^m \le z \le 2^m)$

$M_{III} = M_D + 20 \quad (2^m \le z \le 3^m)$

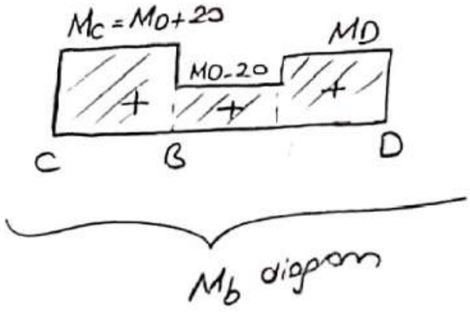
$$\Omega_i = \frac{(M_b)_i\,(L_i)}{G_i\,(I_0)_i}$$

$\Omega_{CD} = 0$ compatibility eq.

$$\Omega_{CD} = \frac{(M_D+20)\,a}{G_C\,(I_0)_C} + \frac{(M_D-20)\,a}{G_S\,(I_0)_S} + \frac{M_D \cdot a}{G_S\,(I_0)_S} = 0$$

$$(I_o)_c = \frac{1}{32}\pi (d_c)^4 = \frac{1}{32}\pi (1{,}2 d_s)^4 \longrightarrow \text{copper}$$

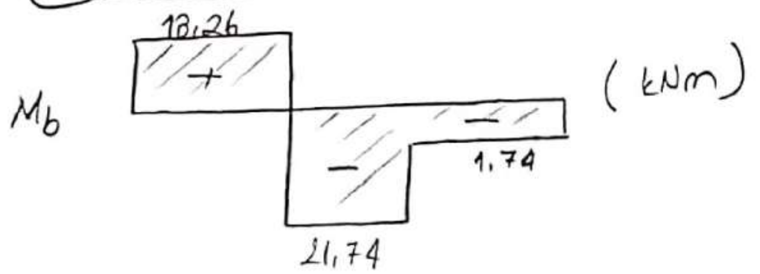
$$(I_o)_s = \frac{1}{32}\pi (d_s)^4 \longrightarrow \text{steel}$$



$$\frac{M_D + 20}{3 \cdot 1{,}2^4} + \frac{M_D - 20}{8} + \frac{M_D}{8} = 0$$

$$M_D = -1{,}74 \text{ kNm}$$

$$M_C = 18{,}26 \text{ kNm}$$



b) $(M_b)_{max} = |-21{,}74|$ kNm $\longrightarrow$ For steel

$(M_b)_{max} = 18{,}26$ kNm $\longrightarrow$ For copper

$$d_s \geq \sqrt[3]{\frac{16\,(M_b)_{max}}{(\tau_{allow})_s\,\pi}} = \sqrt[3]{\frac{16\,(\underbrace{|-21{,}74| \times 10^6}_{Nmm})}{\underbrace{80}_{MPa}\,\pi}} \cong 112 \text{ mm}$$

$$d_c \geq \sqrt[3]{\frac{16\,(M_b)_{max}}{(\tau_{allow})_c\,\pi}} = \sqrt[3]{\frac{16\,(18{,}26 \times 10^6)}{50\,\pi}} \cong 123 \text{ mm}$$

$d_c = 1{,}2 d_s \Longrightarrow$ if $d_s = 112$ mm ; $d_c \cong 135$ mm ✓ $(d_c \geq 123 \text{ mm})$

if $d_s = \underbrace{123}_{d_c}/1{,}2 = d_s \cong 102{,}5$ mm ✗ $(d_s \geq 112 \text{ mm})$