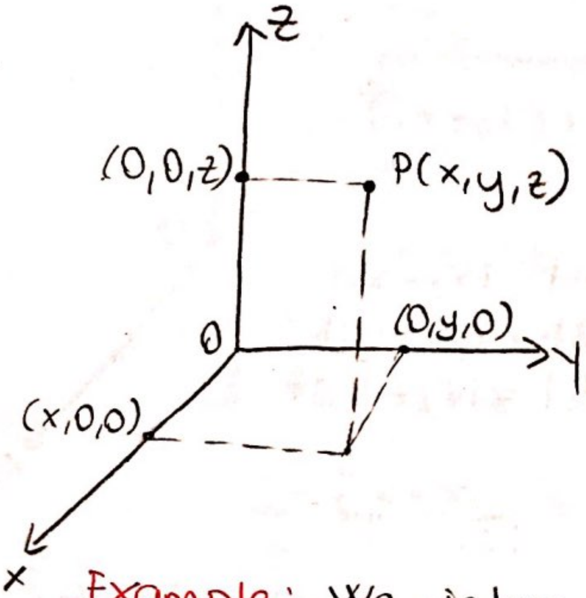
VECTORS AND THE GEOMETRY OF THE SPACE

Three-Dimensional Coordinate Systems



Cartesian coordinates for space are called rectangular coordinates.

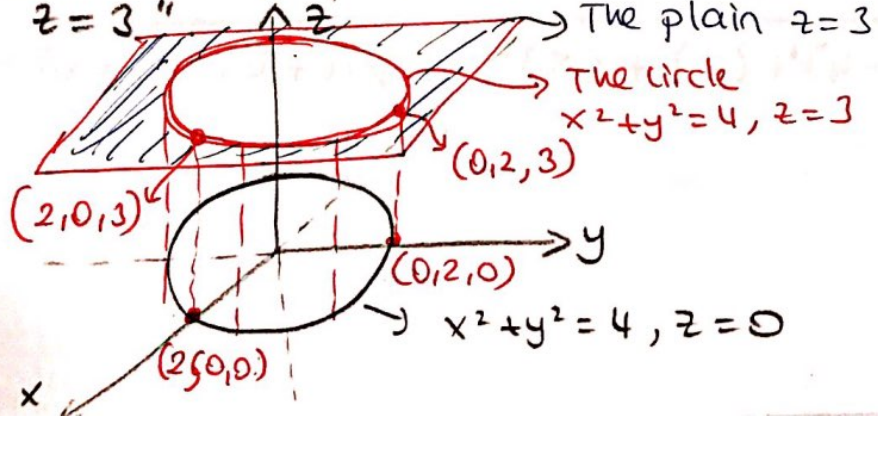
Example: We interpret these equations and inequalities geometrically.

a) $z \geq 0$: The half space consisting of the points on and above the xy-plane.

b) $x = -3$: The plane perpendicular to the x-axis at $x=-3$. This plane lies parallel to the yz-plane and 3 units behind it.

Example: What points $P(x,y,z)$ satisfy the equations $x^2+y^2=4$ and $z=3$?

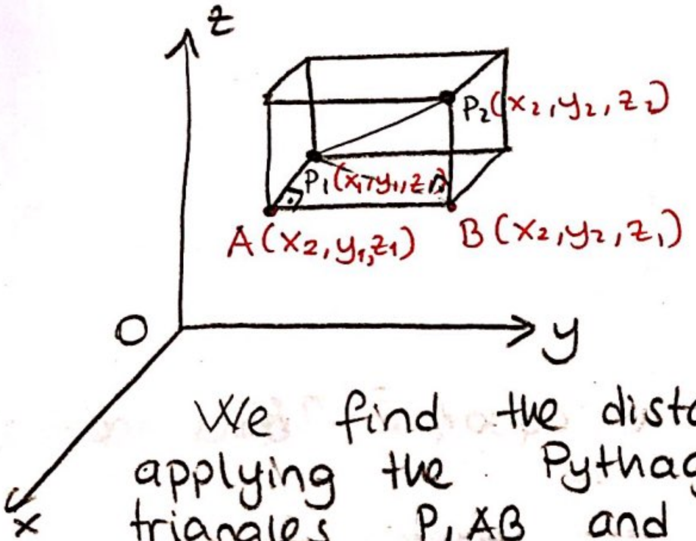
The points lie in the horizontal plane $z=3$ and, in this plane, make up the circle $x^2+y^2=4$. We call this set of points "the circle $x^2+y^2=4$ in the plane $z=3$"



Distance and Spheres in Space:

The distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is;

$$d = |P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$



$|P_1A| = |x_2-x_1|$
$|AB| = |y_2-y_1|$
$|BP_2| = |z_2-z_1|$

We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2.

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

$$|P_1P_2|^2 = (|P_1A|^2 + |AB|^2) + |BP_2|^2$$

$$|P_1P_2|^2 = |x_2-x_1|^2 + |y_2-y_1|^2 + |z_2-z_1|^2$$
$$= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2.$$

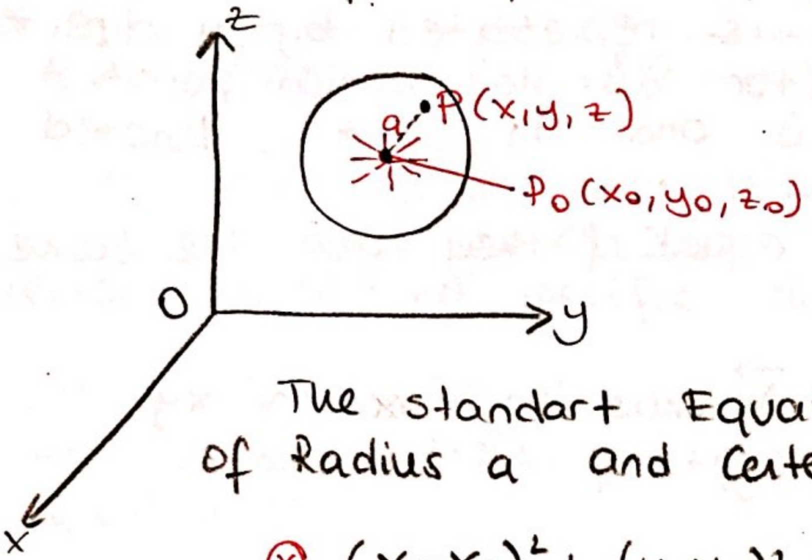
Therefore

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Example: What is the distece between $P_1(2,1,5)$ and $P_2(-2,3,0)$?

$$|P_1P_2| = \sqrt{(-4)^2+(2)^2+(-5)^2} = \sqrt{16+4+25} = \sqrt{45} \approx 6{,}708.$$

We can use the distance formula to write equations for spheres in space;



The standart Equation for the sphere of Radius a and Center (x_0, y_0, z_0);

$$\circledast \quad (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$$

Example: Find the center and radius of the sphere

$$x^2+y^2+z^2+3x-4z+1=0.$$

$$\left(x+\frac{3}{2}\right)^2 + y^2 + (z-2)^2 + 1 - \frac{9}{4} - 4 = 0$$

$$\left(x+\frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

Center $\left(-\frac{3}{2}, 0, 2\right)$ and the radius $r = \frac{\sqrt{21}}{2}$

Example: Find Equation for the sphere whose center and radius is $(1,2,3)$ and $\sqrt{14}$, respectively.

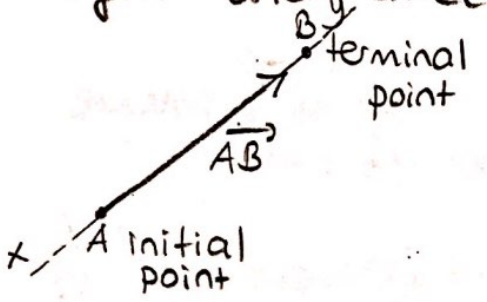
$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14 \quad \text{or}$$

$$x^2+y^2+z^2-2x-4y-6z=0$$

VECTORS

Definition: The vector is represeted by a directed line segment. The vector $\overrightarrow{AB}$ has initial point A and terminal point B and its leght is denoted by $|\overrightarrow{AB}|$.

Two vectors are equal if they have the same lenght and direction.



$\overrightarrow{AB}$ has the direction xy

lenght of $\overrightarrow{AB}$ is $|\overrightarrow{AB}|$

Definition: If $\vec{v}$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the component form of $\vec{v}$ is; $\vec{v} = \langle v_1, v_2 \rangle$

If $\vec{v}$ is a three-dimensional vector, equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of v is; $\vec{v} = \langle v_1, v_2, v_3 \rangle$

⊛ The real numbers v_1, v_2, v_3 are the components of $\vec{v}$.

⊛ Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standart position vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

⊛ If $\vec{v}$ is two dimensional with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as points in the plane, then

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

* The leight (or magnitude) of the vector $\vec{v} = \overrightarrow{PQ}$ is;

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

* The only vector with leght 0 is the zero vector $0 = <0,0>$ or $0 = <0,0,0>$. This is also the only vector with no specific direction.

Example: Find the component form and lenght of the vector with initial point $P(-3,4,1)$ and $Q(-5,2,2)$.

The standart position vector $\vec{v}$ represeting $\overrightarrow{PQ}$ has components;

$$v_1 = x_2 - x_1 = -5+3 = -2$$
$$v_2 = y_2 - y_1 = 2-4 = -2$$
$$v_3 = z_2 - z_1 = 2-1 = 1$$

The component form of $\overrightarrow{PQ}$ is $\vec{v} = <-2,-2,1>$

The leght of $\vec{v}$ is;

$$|\vec{v}| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3$$

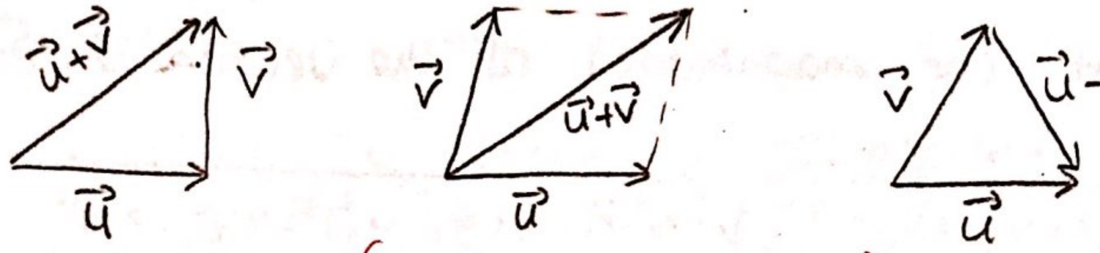
Vector Algebra Operations

Two principal operations involving vectors are vector addition and scalar multiplication.

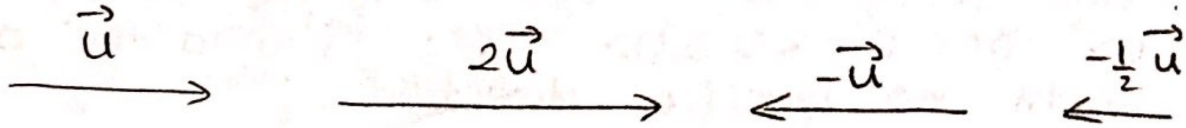
Definition: Let $\vec{u} = <u_1, u_2, u_3>$ and $\vec{v} = <v_1, v_2, v_3>$ be vectors with k, a scalar.

Addition: $\vec{u} + \vec{v} = <u_1+v_1, u_2+v_2, u_3+v_3>$

Scalar Multiplication: $k.\vec{u} = <ku_1, ku_2, ku_3>$



(vector addition)



(Scalar multiples of $\vec{u}$)

Example: Let $\vec{u} = \langle -1, 3, 1 \rangle$ and $\vec{v} = \langle 4, 7, 0 \rangle$. Find the components of;

a) $2\vec{u}+3\vec{v}$ b) $\vec{u}-\vec{v}$ c) $\left|\frac{1}{2}\vec{u}\right|$

a) $2\vec{u}+3\vec{v} = 2\langle -1,3,1\rangle + 3\langle 4,7,0\rangle$

$= \langle -2, 6, 2\rangle + \langle 12, 21, 0\rangle$

$= \langle -2+12, 6+21, 2+0\rangle$

$= \langle 10, 27, 2\rangle$

b) $\vec{u}-\vec{v} = \langle -1,3,1\rangle - \langle 4,7,0\rangle$

$= \langle -1-4, 3-7, 1-0\rangle$

$= \langle -5, -4, 1\rangle$

c) $\left|\frac{1}{2}\vec{u}\right| = \left|\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$

Properties of Vector Operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and a, b be scalars;

1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
3. $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = \vec{0}$
5. $0.\vec{u} = \vec{0}$
6. $1.\vec{u} = \vec{u}$
7. $a(b\vec{u}) = (ab)\vec{u}$
8. $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$
9. $(a+b)\vec{u} = a\vec{u}+b\vec{u}$

Unit Vectors:

A vector $\vec{v}$ of leyht 1 is called a unit vector.
The standart unit vectors are,
$\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

* Any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standart unit vectors as follows;

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

* We call the scalar v_1 the i-component of the vector $\vec{v}$, v_2 the j-component, and v_3 the k-component. In the component form, the vector from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$ is;

$$\overrightarrow{P_1P_2} = (x_2-x_1)\vec{i} + (y_2-y_1)\vec{j} + (z_2-z_1)\vec{k}.$$

* Wherever $\vec{v} \neq 0$, $|\vec{v}|$ is not zero and $\dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\vec{v}$.

Example: Find a unit vector $\vec{u}$ in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

$$\overrightarrow{P_1P_2} = (3-1)\vec{i} + (2-0)\vec{j} + (0-1)\vec{k} = 2\vec{i}+2\vec{j}-\vec{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$$

$$\vec{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{1}{3}(2\vec{i}+2\vec{j}-\vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$$

The vector $\vec{u}$ is in the direction of $\overrightarrow{P_1P_2}$.

* The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point:

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

The Dot Product

Theorem 1 - Angle Between Two Vectors: The angle θ between two nonzero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}| \cdot |\vec{v}|}\right)$$

Definition: The dot product $\vec{u}.\vec{v}$ of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is;

$$\vec{u}.\vec{v} = u_1.v_1 + u_2v_2 + u_3v_3$$

* $\vec{u}.\vec{v} = |\vec{u}|.|\vec{v}|.\cos\theta$

Perpendicular (Orthogonal) Vectors:

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if and only if $\vec{u}.\vec{v} = 0$

Properties of the Dot Product

If $\vec{u}, \vec{v}$ and $\vec{w}$ are any vectors and c is scalar, then

1. $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
2. $(c\vec{u})\vec{v} = \vec{u}.(c\vec{v}) = c(\vec{u}.\vec{v})$
3. $\vec{u}(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$
4. $\vec{u}.\vec{u} = |\vec{u}|^2$
5. $\vec{0}.\vec{u} = 0$

6. $|\vec{u}.\vec{v}| \leq |\vec{u}|.|\vec{v}|$
7. $|\vec{u}+\vec{v}| \leq |\vec{u}|+|\vec{v}|$
8. $\vec{u} \perp \vec{v} \Rightarrow |\vec{u}+\vec{v}|^2 = |\vec{u}|^2+|\vec{v}|^2$

Example: Write the vector $3\vec{i}+\vec{j}$ as a sum of two vectors $\vec{u}$ and $\vec{v}$ such that $\vec{u}$ is parallel to $\vec{i}+\vec{j}$ and $\vec{u}, \vec{v}$ are orthogonal vectors.

Since $\vec{u}$ is parallel to $\vec{i}+\vec{j}$ so, for a scalar t, $\vec{u} = t(\vec{i}+\vec{j})$. $\vec{u}$ and $\vec{v}$ are orthogonal so

$$\vec{u}.\vec{v}=0 \Rightarrow \vec{v}(\vec{i}+\vec{j})=0$$

$$3\vec{i}+\vec{j} = \vec{u}+\vec{v}$$
$$3\vec{i}+\vec{j} = t(\vec{i}+\vec{j})+\vec{v}$$
$$\underbrace{(3\vec{i}+\vec{j})(\vec{i}+\vec{j})} = t(\vec{i}+\vec{j})(\vec{i}+\vec{j}) + \underbrace{\vec{v}(\vec{i}+\vec{j})}_{0}$$
$$3+1 = t\,|\vec{i}+\vec{j}|^2 \qquad \circledast\ |\vec{i}+\vec{j}|^2 = 2$$
$$4 = 2t \Rightarrow t=2$$

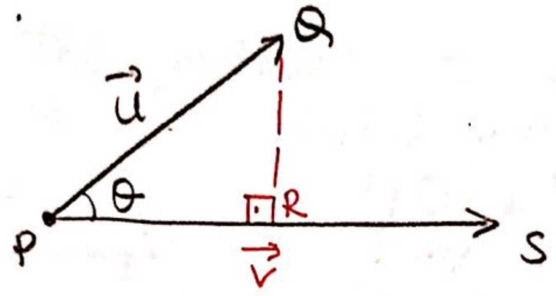
$$\vec{u} = t(\vec{i}+\vec{j}) \Rightarrow \underline{\vec{u} = 2\vec{i}+2\vec{j}}$$

$$\vec{u}+\vec{v} = 3\vec{i}+\vec{j} \Rightarrow \vec{v} = 3\vec{i}+\vec{j} - (2\vec{i}+2\vec{j})$$
$$\underline{\vec{v} = \vec{i}-\vec{j}}$$

Projection

The vector projection of $\vec{u}=\overrightarrow{PQ}$ onto a non-zero vector $\vec{v}=\overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.

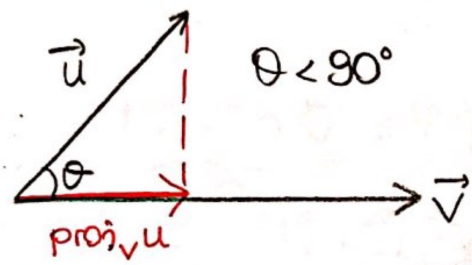
The notation for this vector is $\text{proj}_v u$ (the vector projection of $\vec{u}$ onto $\vec{v}$)



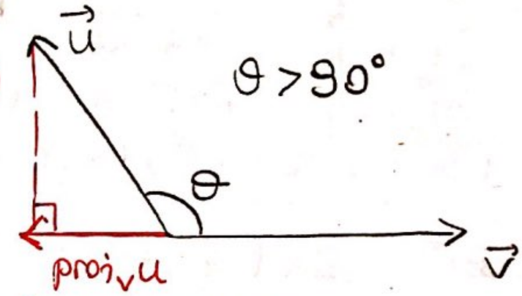
(*) $\text{proj}_v u = (|\vec{u}|.\cos\theta).\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u}.\vec{v}}{|\vec{v}|}.\dfrac{\vec{v}}{|\vec{v}|} = \left(\dfrac{\vec{u}.\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

(*) The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is the scalar;

$$|\vec{u}|.\cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{v}|} = \vec{u}.\ \frac{\vec{v}}{|\vec{v}|}.$$

(*) 

Lenght of $\text{proj}_v u \Rightarrow |\vec{u}|\cos\theta$ and direction $\dfrac{\vec{v}}{|\vec{v}|}$

(*) 

Lenght of the $\text{proj}_v u \Rightarrow -|\vec{u}|\cos\theta$ and direction $-\dfrac{\vec{v}}{|\vec{v}|}$.

Example: Find the vector projection of $\vec{u} = 6\vec{i}+3\vec{j}+2\vec{k}$ onto $\vec{v} = \vec{i}-2\vec{j}-2\vec{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

$$\text{Proj}_v u = \left(\frac{\vec{u}.\vec{v}}{|\vec{v}|^2}\right).\vec{v} = \frac{\cancel{6+(-6)}+(-4)}{(1+4+4)}(\vec{i}-2\vec{j}-2\vec{k})$$

$$\text{Proj}_v u = -\frac{4}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{8}{9}\vec{k}$$

Scalar component of $\vec{u}$ in the direction of $\vec{v}$;

$$|\vec{u}|.\cos\theta = \vec{u}.\frac{\vec{v}}{|\vec{v}|} = (6\vec{i}+3\vec{j}+2\vec{k}).\left(\frac{1}{3}\vec{i}-\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)$$

$$= 6.\frac{1}{3} + 3.\left(-\frac{2}{3}\right) + 2.\left(-\frac{2}{3}\right)$$

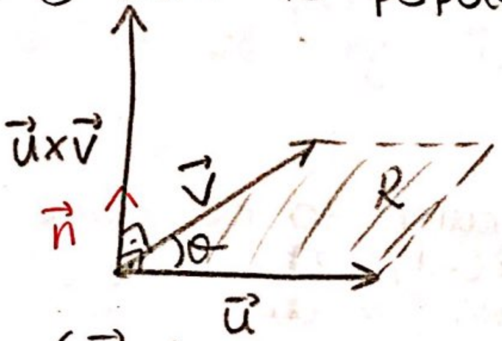
$$= 2 - 2 - \frac{4}{3} = -4/3.$$

Cross Product:

Let $\vec{u}$ and $\vec{v}$ be non zero vectors in space. If $\vec{u}$ and $\vec{v}$ are not parallel, they determine a plane. The cross product $\vec{u}\times\vec{v}$ is the vector defined as follows;

$$\vec{u}\times\vec{v} = (|\vec{u}|.|\vec{v}|.\sin\theta)\vec{n}$$

⊛ $\vec{u}\times\vec{v}$ is perpendicular to the plane.



The vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, b/c it's a scalar multiple of $\vec{n}$.

($\vec{n}$ is a unit vector)

$$|\vec{u}\times\vec{v}| = |\vec{u}|.|\vec{v}|.\sin\theta = \text{Area}(R)$$

⊛ Unlike the dot product, the cross product is a vector.

Parallel vectors: Non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u}\times\vec{v} = \vec{0}$.

Properties of the Cross Product:
If u, v, w are any vectors and r, s are scalars, the

1. $(r\vec{u})\times(s\vec{v}) = (rs)(\vec{u}\times\vec{v})$
2. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
3. $\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})$
4. $(\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u} + \vec{w}\times\vec{u}$
5. $\vec{0}\times\vec{u} = \vec{0}$
6. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}.\vec{w})\vec{v} - (\vec{u}.\vec{v}).\vec{w}$ ⊛ $\vec{u}\times(\vec{v}\times\vec{w}) \neq (\vec{u}\times\vec{v})\times\vec{w}$
7. $(\vec{u}\times\vec{v}).\vec{u} = 0$ and $(\vec{u}\times\vec{v}).\vec{v} = 0$.

8. $\vec{u} \times \vec{u} = \vec{0}$.

9. $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$

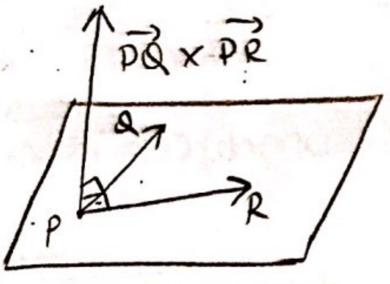
Calculating the Cross Product as a Determinant:

If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example: Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane b/c it's perpendicular to both vectors.



$$\vec{PQ} = (2-1)\vec{i} + (1+1)\vec{j} + (-1-0)\vec{k} = \vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{PR} = (-1-1)\vec{i} + (1+1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6\vec{i} + 0.\vec{j} + 6\vec{k} = 6\vec{i} + 6\vec{k}$$

Triple Scalar or Box Product

The product $(\vec{u} \times \vec{v}) . \vec{w}$ is called the triple scalar product of $\vec{u}, \vec{v}$ and $\vec{w}$.

$$|(\vec{u} \times \vec{v}) . \vec{w}| = |\vec{u} \times \vec{v}| . |\vec{w}| . \cos\theta$$

The absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by $\vec{u}, \vec{v}$ and $\vec{w}$.

Calculating the Triple Scalar Product as a Determinant

$$(\vec{u} \times \vec{v}) . \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

⊛ $\vec{u} . (\vec{v} \times \vec{w}) = \vec{v} . (\vec{w} \times \vec{u}) = \vec{w} (\vec{u} \times \vec{v})$

⊛ $\vec{u} . (\vec{v} \times \vec{w}) = - \vec{u} . (\vec{w} \times \vec{v})$

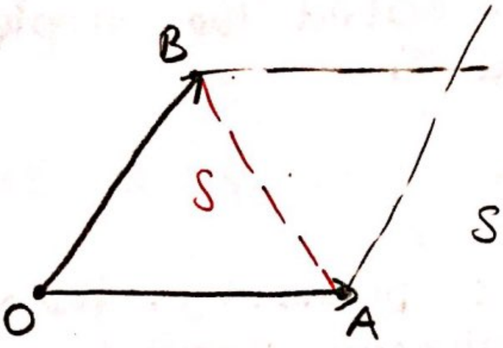
⊛ If $\vec{u} . (\vec{v} \times \vec{w}) = 0$, then these vectors are on the same plane.

Example: Find the volume of the box (parallelepiped) determined by $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = -2\vec{i} + 3\vec{k}$ and $\vec{w} = 7\vec{j} - 4\vec{k}$.

$$V = (\vec{u} \times \vec{v}) . \vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23$$

The volume is $|(\vec{u} \times \vec{v}) . \vec{w}| = |-23| = 23$ units cubed.

Example: Find the formula, which gives the area of the triangle which has cusp points $O(0,0)$, $A(a_1,a_2)$, $B(b_1,b_2)$



$$\vec{OA} = a_1\vec{i} + a_2\vec{j}$$
$$\vec{OB} = b_1\vec{i} + b_2\vec{j}$$

$$S = \text{Area}(OAB) = \frac{1}{2}\left|\vec{OA} \times \vec{OB}\right|$$

$$S = \frac{1}{2}\left|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix}\right| = \frac{1}{2}\cdot\left|0\vec{i} + 0\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\right|$$

$$S = \frac{1}{2}\left|(a_1b_2 - a_2b_1)\vec{k}\right|$$

$$S = \frac{1}{2}\sqrt{(a_1b_2 - a_2b_1)^2} \Rightarrow S = \frac{1}{2}\left|a_1b_2 - a_2b_1\right|$$

Example: Prove that the points $A(1,1,3)$, $B(2,1,2)$, $C(-1,-2,8)$ and $D(2,0,3)$ are on the same plane.

If these points are on the same plane so the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are on the same plane. Therefore the triple scalar product of them must be zero.

$$\vec{AB} = (2-1)\vec{i} + (1-1)\vec{j} + (2-3)\vec{k} = \vec{i} - \vec{k}$$
$$\vec{AC} = (-1-1)\vec{i} + (-2-1)\vec{j} + (8-3)\vec{k} = -2\vec{i} - 3\vec{j} + 5\vec{k}$$
$$\vec{AD} = (2-1)\vec{i} + (0-1)\vec{j} + (3-3)\vec{k} = \vec{i} - \vec{j}$$

$$\vec{AB}\cdot(\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & 0 & -1 \\ -2 & -3 & 5 \\ 1 & -1 & 0 \end{vmatrix} = 5 - 5 = 0.$$