★ Solve the system of differential equations

$$\frac{dy}{dx} = 2y - z + 1 \quad (1)$$

$$\frac{dz}{dx} = y + e^{2x} \quad (2) \quad \text{using Elimination method.}$$

$$(D-2)\,y + z = 1$$

$$D-2 \qquad -y + Dz = e^{2x}$$

$$+ \overline{\qquad\qquad\qquad\qquad}$$

$$(1 + D^2 - 2D)\,z = 1 + 2e^{2x} - 2e^{2x} \Rightarrow (D-1)^2 z = 1 \quad \left(z'' - 2z' + z = 1\right)$$

$$(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \quad \Rightarrow z_h = c_1 e^x + c_2 x e^x$$

$$z_p = a \,,\; z_p' = z_p'' = 0 \quad \Big\} \quad 0 - 0 + a = 1 \Rightarrow a = 1$$

$$\boxed{z = c_1 e^x + c_2 x e^x + 1}$$

$$y = Dz - e^{2x} = c_1 e^x + c_2 e^{2x} + 2c_2 x e^{2x} - e^{2x} = \boxed{(c_1 + c_2 + 2c_2 x)e^{2x} - e^{2x}}$$

★ Using the Derivation and elimination, solve the system

$$\frac{dx}{dt} + 2y - x = 0 \qquad (D-1)\,x + 2y = 0$$

$$\frac{dy}{dt} + y - x = -\sin t \qquad (D-1) \quad -x + (D+1)\,y = -\sin t$$

$$+ \overline{\qquad\qquad\qquad\qquad}$$

$$(D^2 - 1 + 2)\,y = -\cos t + \sin t \quad \left[\, y'' + y = -\cos t + \sin t \,\right]$$

$$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i \qquad y_h = c_1 \cos t + c_2 \sin t$$

$$y_p = (A\cos t + B\sin t)\,t$$

$$y_p' = (-A\sin t + B\cos t)\,t + (A\cos t + B\sin t)$$

$$y_p'' = (-A\cos t - B\sin t)\,t + 2(-A\sin t + B\cos t)$$

$$(\cancel{-A\cos t} - B\sin t)\,t + 2(-A\sin t + B\cos t) + \cancel{(A\cos t + B\sin t)\,t} = -\cos t + \sin t$$

$$A = -\frac{1}{2} \qquad B = -\frac{1}{2}$$

$$y_p = -\frac{1}{2}t(\cos t + \sin t)$$

$$\boxed{y = c_1 \cos t + c_2 \sin t - \frac{t}{2}(\cos t + \sin t)} \quad x = c_1(\cos t - \sin t) + c_2(\cos t + \sin t) - \frac{\cos t}{2} + \frac{\sin t}{2} - t\cos t$$

* solve the system $\left.\begin{aligned} 2\frac{dx}{dt}+\frac{dy}{dt}+x+5y=4t \\ \frac{dx}{dt}+\frac{dy}{dt}+2x+2y=2 \end{aligned}\right\}$ using determinant method.

$$\left.\begin{aligned}(2D+1)x+(D+5)y=4t \\ (D+2)x+(D+2)y=2\end{aligned}\right\} \quad \Delta=\begin{vmatrix}2D+1 & D+5 \\ D+2 & D+2\end{vmatrix}=D^2-2D-8$$

$$x=\frac{\Delta_1}{\Delta}=\frac{\begin{vmatrix}4t & D+5 \\ 2 & D+2\end{vmatrix}}{D^2-2D-8} \Rightarrow (D^2-2D-8)x=8t-6$$

$$\left.\begin{aligned}r^2-2r-8=0 \\ -4 \quad +2\end{aligned}\right\} \begin{aligned}r_1=4 \\ r_2=-2\end{aligned} \qquad x_h=c_1e^{4t}+c_2e^{-2t}$$

$$x_p=at+b,\ x_p'=a,\ x_p''=0 \left.\begin{aligned}\Big\}\ -2a-8at-8b=8t-6 \\ \boxed{a=-1}\ -2a-8b=-6 \\ b=1\end{aligned}\right\} x_p=-t+1$$

$$\boxed{x=c_1e^{4t}+c_2e^{-2t}-t+1}$$

$$y=\frac{\Delta_2}{\Delta}=\frac{\begin{vmatrix}2D+1 & 4t \\ D+2 & 2\end{vmatrix}}{D^2-2D-8} \Rightarrow (D^2-2D-8)y=-8t-2$$

$$y_h=c_3e^{4t}+c_4e^{-2t}$$

$$y_p=at+b,\ y_p'=a,\ y_p''=0 \left.\begin{aligned}\Big\}\ -2a-8at-8b=-8t-2 \\ a=1,\ b=0\end{aligned}\right\} y_p=t$$

$$y=c_3e^{4t}+c_4e^{-2t}+t$$

$$4c_1e^{4t}-2c_2e^{-2t}\cancel{-1}+4c_3e^{4t}-2c_4e^{-2t}\cancel{+1}+2c_1e^{4t}+2c_2e^{-2t}\cancel{-2t}\cancel{+2}+2c_3e^{4t}+2c_4e^{-2t}$$
$$\cancel{+2t}=\cancel{2}$$

$$\underbrace{[4c_1+4c_3+2c_1+2c_3]}_{=0}e^{4t}+\underbrace{[-2c_2-2c_4+2c_2+2c_4]}_{c_2=c_4}e^{-2t}=0$$

$c_3=-c_1$

?

★ Find the solution of the differential equation $y''+16y=5\sin x$ which possesses the initial conditions $y'(0)=y(0)=0$, by using the Laplace Transform.

$$L\{y''\}+16L\{y\}=L\{5\sin x\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{0}-\underbrace{y'(0)}_{0}+16Y(s)=\frac{5}{s^2+1}$$

$$(s^2+16)Y(s)=\frac{5}{s^2+1} \Rightarrow Y(s)=\frac{5}{(s^2+1)(s^2+16)}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)}=5L^{-1}\underbrace{\left\{\frac{1}{(s^2+1)(s^2+16)}\right\}} \longrightarrow \left(\frac{1}{s^2+1}-\frac{1}{s^2+16}\right)\frac{1}{15}$$

$$\frac{5}{15}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\}-L^{-1}\left\{\frac{1}{s^2+16}\right\}\right]$$

$$y(t)=\frac{1}{3}\left[\sin x-\frac{1}{4}\sin 4x\right]$$

★ solve the initial value problem $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^{-t}$ $y(0)=1$, $y'(0)=2$ using Laplace and inverse laplace transform.

$$L\{y''\}+2L\{y'\}+L\{y\}=L\{te^{-t}\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{2}+2sY(s)-2\underbrace{y(0)}_{1}+Y(s)=\frac{1}{(s+1)^2}$$

$$(s^2+2s+1)Y(s)=\frac{1}{(s+1)^2}+(s+4) \Rightarrow Y(s)=\frac{1}{(s+1)^4}+\frac{s+4}{(s+1)^2}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)}=\underbrace{L^{-1}\left\{\frac{1}{(s+1)^4}\right\}}_{\frac{t^3}{3!}e^{-t}}+\underbrace{L^{-1}\left\{\frac{s+1}{(s+1)^2}\right\}}_{e^{-t}}+\underbrace{L^{-1}\left\{\frac{3}{(s+1)^2}\right\}}_{3te^{-t}}$$

$$y(t)=\frac{t^3}{3!}e^{-t}+e^{-t}+3te^{-t}$$

★ Evaluate the integral $\int_0^\infty e^{-st}(1+t\sin t)\,dt$ by using Laplace transform definition and properties of Laplace transform.

$$\int_0^\infty e^{-st}(1+t\sin t)\,dt = L\{1+t\sin t\} = L\{1\}+L\{t\sin t\}$$

$$= \frac{1}{s} + (-1)\frac{d}{ds}L\{\sin t\} = \frac{1}{s} - \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{s} + \frac{2s}{(s^2+1)^2}$$

★ Using Laplace transform, find the solution of the initial value problem $y''+2y'+5y=0$, $y(0)=2$, $y'(0)=-1$

$$s^2Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{-1} + 2[sY(s) - \underbrace{y(0)}_{2}] + 5Y(s) = 0$$

$$(s^2+2s+5)Y(s) = 2s+3 \Rightarrow Y(s) = \frac{2s+3}{s^2+2s+5}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{} = L^{-1}\left\{\frac{2s+3}{s^2+2s+5}\right\}$$

$$y(t) = L^{-1}\left\{\frac{2s+3}{(s+1)^2+4}\right\} = 2L^{-1}\left\{\frac{s+\frac{3}{2}}{(s+1)^2+4}\right\} = 2\left\{\frac{s+1}{(s+1)^2+4}\right\} + 2\left\{\frac{\frac{1}{2}}{(s+1)^2+4}\right\}$$

$$y(t) = 2e^{-t}\cos 2t + e^{-t}\frac{\sin 2t}{2}$$

☆ Using laplace transform solve the initial value problem

$y(0)=1$, $y'(0)=-1$, $y''-y'-6y=0$

$$s^2Y(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{-1}-[sY(s)-\underbrace{y(0)}_{1}]-6Y(s)=0$$

$$(s^2-s-6)Y(s)=s-2 \Rightarrow Y(s)=\frac{s-2}{s^2-s-6}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)}=L^{-1}\left\{\frac{s-2}{s^2-s-6}\right\}$$

$$\frac{A}{s-3}+\frac{B}{s+2}=\frac{s-2}{s^2-s-6} \quad \left\{\begin{array}{l} A+B=1 \\ 2A-3B=-2 \end{array}\right\} \quad \begin{array}{l} A=\frac{1}{5} \\ B=\frac{4}{5} \end{array}$$

$$y(t)=\frac{1}{5}\underbrace{L^{-1}\left\{\frac{1}{s-3}\right\}}_{e^{3t}}+\frac{4}{5}\underbrace{L^{-1}\left\{\frac{1}{s+2}\right\}}_{e^{-2t}}$$

☆ Using Laplace transform, solve $y''+w^2y=\cos 2t$, $y(0)=1$, $y'(0)=0$
$(w^2\neq 4)$

$$s^2Y(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{0}+w^2Y(s)=\frac{s}{s^2+4}$$

$$(s^2+w^2)Y(s)=\frac{s}{s^2+4}+s \Rightarrow Y(s)=\frac{s}{(s^2+4)(s^2+w^2)}+\frac{s}{s^2+w^2}$$

$$L^{-1}\{Y(s)\}=L^{-1}\left\{\frac{s}{(s^2+4)(s^2+w^2)}\right\}+L^{-1}\left\{\frac{s}{s^2+w^2}\right\}$$

$$y(t)=\frac{1}{4-w^2}L^{-1}\left\{\frac{s}{(s^2+w^2)}-\frac{s}{(s^2+4)}\right\}+L^{-1}\left\{\frac{s}{s^2+w^2}\right\}$$

$$y(t)=\frac{1}{4-w^2}\left[\cos wt-\cos 2t\right]+\cos wt$$

④ Using Laplace transform solve the initial value problem

$$y'' + 4y' + 5y = 10e^{t} \quad , \quad y(0) = y'(0) = 0.$$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 4[sY(s) - \underbrace{y(0)}_{0}] + 5Y(s) = 10\frac{1}{s-1}$$

$$(s^2+4s+5)Y(s) = \frac{10}{s-1} \Rightarrow Y(s) = \frac{10}{(s-1)(s^2+4s+5)}$$

$$L^{-1}\{Y(s)\} = 10\, L^{-1}\left\{\frac{1}{(s-1)(s^2+4s+5)}\right\}$$

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+4s+5} = \frac{1}{(s-1)(s^2+4s+5)} \Rightarrow \left.\begin{array}{l} A+B=0 \\ 4A-B+C=0 \\ 5A-C=1 \end{array}\right\} \begin{array}{l} A = \frac{1}{10} \\ B = -\frac{1}{10} \\ C = -\frac{1}{2} \end{array}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{-s-5}{s^2+4s+5}\right\}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{(s+2)+3}{(s+2)^2+1}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{(s+2)}{(s+2)^2+1}\right\} - 3L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$y(t) = e^{t} - e^{-2t}\cos t - 3e^{-2t}\sin t$$

* Using laplace transform, solve $y''+2y'+y=4e^{-t}$ $\quad y(0)=2,\ y'(0)=-1$

$$s^2Y(s)-s\underbrace{y(0)}_{2}-\underbrace{y'(0)}_{-1}+2[sY(s)-\underbrace{y(0)}_{2}]+Y(s)=\frac{4}{s+1}$$

$$(s^2+2s+1)Y(s)=\frac{4}{s+1}+2s+3 \Rightarrow Y(s)=\frac{4}{(s+1)^3}+\frac{2s+3}{(s+1)^2}$$

$$\underbrace{L^{-1}\{Y(s)\}}=4L^{-1}\left\{\frac{1}{(s+1)^3}\right\}+L^{-1}\left\{\frac{2s+3}{(s+1)^2}\right\}$$

$$y(t)=4e^{-t}\frac{t^2}{2!}+2L^{-1}\left\{\frac{s+\frac{3}{2}}{(s+1)^2}\right\}=4e^{-t}\frac{t^2}{2!}+2L^{-1}\left\{\frac{(s+1)}{(s+1)^2}\right\}+2L^{-1}\left\{\frac{\frac{1}{2}}{(s+1)^2}\right\}$$

$$y(t)=4e^{-t}\frac{t^2}{2!}+2e^{-t}+e^{-t}.t$$

Solve the system $\left.\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} - 2x - 4y &= e^{t} \\ \frac{dx}{dt} + \frac{dy}{dt} - y &= e^{4t} \end{aligned}\right\}$ using determinant method.

$$\left.\begin{aligned} (D-2)x + (D-4)y &= e^{t} \\ Dx + (D-1)y &= e^{4t} \end{aligned}\right\} \quad \Delta = \begin{vmatrix} D-2 & D-4 \\ D & D-1 \end{vmatrix} = (D+2)$$

$$x = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} e^{t} & D-4 \\ e^{4t} & D-1 \end{vmatrix}}{(D+2)} \Rightarrow (D+2)x = 0 \Rightarrow r+2=0 \Rightarrow r_1 = -2 \quad \boxed{x = c_1 e^{-2t}}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} D-2 & e^{t} \\ D & e^{4t} \end{vmatrix}}{(D+2)} \Rightarrow (D+2)y = 2e^{4t} - e^{t}$$

$y_h = c_2 e^{-2t}$

$$\left.\begin{aligned} y_{p1} &= ke^{4t} \\ y'_{p1} &= 4ke^{4t} \end{aligned}\right\} \quad \begin{aligned} 4k+2k &= 2 \\ k &= \frac{1}{3} \end{aligned} \qquad y_{p1} = \frac{1}{3}e^{4t}$$

$$\left.\begin{aligned} y_{p2} &= ke^{t} \\ y'_{p2} &= ke^{t} \end{aligned}\right\} \quad \left.\begin{aligned} k+2k &= -1 \\ k &= -\frac{1}{3} \end{aligned}\right\} \quad y_{p2} = -\frac{1}{3}e^{t}$$

$$y = c_2 e^{-2t} + \frac{1}{3}e^{4t} - \frac{1}{3}e^{t}$$

$$-2c_1e^{-2t} - 2c_2e^{-2t} + \cancel{\frac{4}{3}e^{4t}} - \cancel{\frac{1}{3}e^{t}} - c_2e^{-2t} - \cancel{\frac{1}{3}e^{4t}} + \cancel{\frac{1}{3}e^{t}} = \cancel{e^{4t}}$$

$$(-2c_1 - 3c_2)e^{-2t} = 0 \Rightarrow \boxed{c_2 = -\frac{2c_1}{3}}$$