Exercises (1)

1) Find the rate of change of the function $f(x,y)=x^2y$ at the point $(-1,-1)$ in the direction of the vector $u=i+2j$.

2) Find the rate of change of the function $f(x,y)=x^2+y^2$ at the point $(1,-2)$ in the direction making a positive angle of $60°$ with the positive x-axis.

3) Find the gradient of the function $f(x,y)=e^{xy}$ at $(2,0)$.

4 Find the directional derivative of $f(x,y,z)=x^2y+y^2z+z^2x$ at $(1,-1,1)$ in the direction $u=i+j+2k$.

Systems of Equations

Let
$$F(x,y,z,w)=0$$
$$G(x,y,z,w)=0.$$

Let $\begin{cases} x=x(z,w) \\ y=y(z,w) \end{cases}$, $\begin{cases} x=x(y,w) \\ \cancel{y}\,z=z(y,w) \end{cases}$, $\begin{cases} x=x(y,z) \\ w=w(y,z) \end{cases}$

$\begin{cases} y=y(x,w) \\ z=z(x,w) \end{cases}$, $\begin{cases} y=y(x,z) \\ w=w(x,z) \end{cases}$, $\begin{cases} z=z(x,y) \\ w=w(x,y) \end{cases}$.

$$\left(\frac{\partial x}{\partial z}\right)_w \Rightarrow \quad x = x(z,w), \quad y = y(z,w)$$

$$\left(\frac{\partial x}{\partial z}\right)_y \Rightarrow \quad x = x(y,z), \quad w = \cancel{y}(y,z)$$

Example -

Given the equations $F(x,y,z,w)=0$, $G(x,y,z,w)=0$ where F and G have continuous first partial derivatives, calculate $\left(\frac{\partial x}{\partial z}\right)_w = ?$

Solution. Diff. w.r.t. z

$$F_1 \cdot \frac{\partial x}{\partial z} + F_2 \cdot \frac{\partial y}{\partial z} + F_3 = 0$$

$$G_1 \frac{\partial x}{\partial z} + G_2 \cdot \frac{\partial y}{\partial z} + G_3 = 0.$$

$$\begin{cases} x = x(z,w) \\ y = y(z,w) \end{cases}$$

$$\left(\frac{\partial x}{\partial z}\right)_w = - \frac{F_3 G_2 - F_2 G_3}{F_1 G_2 - F_2 G_1}$$

by cramer's rule.

Example. Let x, y, u and v be related by the equations

$$u = x^2 + xy - y^2$$
$$v = 2xy + y^2 .$$

Find (a) $\left(\frac{\partial x}{\partial u}\right)_v$, b) $\left(\frac{\partial x}{\partial u}\right)_y$

at the point $(x, y) = (2, -1)$.

Solution.

a) $\left(\frac{\partial x}{\partial u}\right)_v \Rightarrow x = x(u, v)$
$y = y(u, v)$.

Diff. w.r.t. u,

$$1 = \frac{\partial u}{\partial u} = (2x+y)\frac{\partial x}{\partial u} + (x-2y)\frac{\partial y}{\partial u}$$
$$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x+2y)\frac{\partial y}{\partial u}.$$

At $(2, -1)$,

$$1 = 3\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u}$$
$$0 = -2\frac{\partial x}{\partial u} + 2\frac{\partial y}{\partial u}$$

$\Rightarrow \frac{\partial x}{\partial u} = \frac{1}{7}$ by cramer's rule or eliminating

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b) $\left(\frac{\partial x}{\partial u}\right)_y = ?$ $\Rightarrow$ $x = x(u,y)$
$v = v(u,y)$.

Diff. w.r.t. u,

$1 = \frac{\partial u}{\partial u} = (2x+y)\frac{\partial x}{\partial u}$

$\frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u}$.

$x=2,\ y=-1 \Rightarrow 1 = 3\frac{\partial x}{\partial u}$

$\Rightarrow \frac{\partial x}{\partial u} = 1/3$.

Jacobian Determinants

Definition. The Jacobian determinant or Jacobian of the two functions $u=u(x,y)$, $v=v(x,y)$ is the determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

Let $F(x,y,\ldots)$, $G(x,y,\ldots)$.

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}$$

Let $F(x, y, z, \ldots)$, $G(x, y, z, \ldots)$
$H(x, y, z, \ldots)$

$$\frac{\partial(F, G, H)}{\partial(x, y, z)} = \begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}$$

<u>Example.</u>

Let $F(x, y, z, w) = 0$,
$G(x, y, z, w) = 0$.

$$\left(\frac{\partial x}{\partial z}\right)_w = ?$$

$$\left(\frac{\partial x}{\partial z}\right)_w = - \frac{\dfrac{\partial(F, G)}{\partial(z, y)}}{\dfrac{\partial(F, G)}{\partial(z, y)}}.$$

The Implicit Function Theorem.

Consider a system of n equations in $n+m$ variables

$$F_{(1)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0$$

$$F_{(2)}(x_1, x_2, \ldots, x_m, y_1, \ldots, y_n) = 0$$

$$\vdots$$

$$F_{(n)}(x_1, x_2, \ldots, x_m, y_1, \ldots, y_n) = 0.$$

and a point $P_0 = (a_1, a_2, \ldots, a_m, b_1, \ldots, b_n)$ that satisfies the system.

Suppose each of the functions $F_{(i)}$ has continuous first partial derivatives w.r.t. each of the variables x_j, y_k ($i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, n$) near P_0. Also, suppose

$$\left.\frac{\partial(F_{(1)}, F_{(2)}, \ldots, F_{(n)})}{\partial(y_1, y_2, \ldots, y_n)}\right|_{P_0} \neq 0$$

Then the system can be solved for $y_1, y_2, \dots, y_n$ as functions of $x_1, x_2, \dots, x_m$ near P_0.

That is,

$$\phi_1(x_1, \dots, x_m), \dots, \phi_n(x_1, \dots, x_m)$$

such that

$$\phi_j(a_1, \dots, a_m) = b_j, \qquad j = 1, \dots, n,$$

and such that

$$F_{(1)}(x_1, x_2, \dots, x_m, \phi_1(x_1, \dots, x_m), \dots, \phi_n(x_1, \dots, x_m)) = 0,$$

$$\vdots$$

$$F_{(n)}(x_1, \dots, x_m, \phi_1(x_1, \dots, x_m), \dots, \phi_n(x_1, \dots, x_m)) = 0$$

hold for all $(x_1, \dots, x_m)$ sufficiently near $(a_1, \dots, a_m)$.

Moreover,

$$\frac{\partial \phi_i}{\partial x_j} = \left(\frac{\partial \phi_i}{\partial x_j}\right)_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m}$$

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$$\frac{\partial \phi_i}{\partial x_j} = - \frac{\dfrac{\partial(F_{(1)}, F_{(2)}, \dots, F_{(n)})}{\partial(y_1, \dots, x_j, \dots, y_n)}}{\dfrac{\partial(F_{(1)}, F_{(2)}, \dots, F_{(n)})}{\partial(y_1, y_2, \dots, y_n)}}$$

<u>Example</u>-

Show that the system

$$xy^2 + xzu + yv^2 = 3$$
$$x^3yz + 2xv - u^2v^2 = 2$$

can be solved for (u, v) as a function of (x, y, z) near the point P_0 where $(x, y, z, u, v) = (1,1,1,1,1)$ and find $\frac{\partial u}{\partial y}$ for the solution at $(x, y, z) = (1,1,1)$.

<u>Solution</u>- $F(x, y, z, u, v) = xy^2 + xzu + yv^2 - 3$

$G(x, y, z, u, v) = x^3yz + 2xv - u^2v^2 - 2$

$$\Rightarrow |J| = \frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} xz & 2yv \\ -2uv^2 & 2x - 2u^2v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4$$

$J \neq 0$ By implicit Function th the equations can be solved for u, v as functions of x, y, z).

$$\left.\frac{\partial(F,G)}{\partial(u,y)}\right|_{P_0} = \left|\begin{matrix} xz & 2xy+v^2 \\ -2uv^2 & x^3z \end{matrix}\right|_{P_0}$$

$$= \left|\begin{matrix} 1 & 3 \\ -2 & 1 \end{matrix}\right| = 7.$$

$$\left(\frac{\partial v}{\partial y}\right)_{x,z} = -\frac{\left.\frac{\partial(F,G)}{\partial(u,y)}\right|_{P_0}}{\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}} = -7/4$$

Property. Chain Rule.

$$\frac{\partial(z_1,\dots,z_n)}{\partial(x_1,\dots,x_n)} = \frac{\partial(z_1,\dots,z_n)}{\partial(y_1,\dots,y_n)} \cdot \frac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}$$

$\Rightarrow$

$$\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)} = \frac{1}{\frac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}}$$