(1)

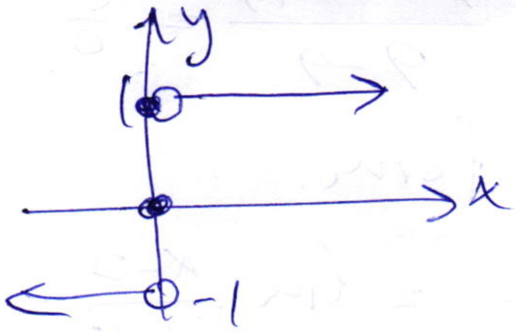
Example. Show that $\lim\limits_{x \to x_0} |x| = |x_0|$, $x_0 \in \mathbb{R}$.

Solution

Given any $\varepsilon > 0$ find $\delta > 0$ such that $||x| - |x_0|| < \varepsilon$ whenever $0 < |x - x_0| < \delta$.

Since $||x| - |x_0|| \leq |x - x_0| < \varepsilon$ by choosing $\delta = \varepsilon$.

Example. Let $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$



$\lim\limits_{x \to 0} f(x)$ does not exist.

Theorem. If $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a} g(x) = M$, then

1) $\lim\limits_{x \to a} [f(x) + g(x)] = L + M$,

2) $\lim\limits_{x \to a} [f(x) - g(x)] = L - M$,

3) $\lim\limits_{x \to a} f(x) \cdot g(x) = L \cdot M$

4) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

(2)

5) If $f(x) \le g(x)$ on an interval containing a in its interior, then

$$L \le M.$$

6) If m is an integer, n is a ~~positive~~ natural number, then

$$\lim_{x \to a} [f(x)]^{g(x)} = L^{m/n}, \text{ provided } L > 0$$

if n is even and $L \neq 0$ if $m \cancel{<} 0$.

Example. Evaluate

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x^2 - 9} = \frac{9 - 18 + 9}{9 - 9} = \frac{0}{0}$$

indeterminate form.

$$\lim_{\substack{x \to 3 \\ x \neq 3}} \frac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}(x+3)} = \lim_{x \to 3} \frac{x-3}{x+3} = \frac{0}{6} = 0$$

Theorem ① If $P(x)$ is a polynomial function and $a \in \mathbb{R}$, then

$$\lim_{x \to a} P(x) = P(a).$$

② If $P(x)$ and $Q(x)$ are polynomial and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

Theorem. (The Squeeze theorem)

Suppose that $f(x) \le g(x) \le h(x)$ holds for all x in some open interval containing a except possibly at $x = a$ itself.

Suppose also that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then $\lim_{x \to a} g(x) = L$ also.

Example. Given that

$3 - x^2 \le f(x) \le 3 + x^2$ for all $x \ne 0$,

Find $\lim_{x \to 0} f(x) = ?$

Solution. Since $\lim_{x \to 0} 3 - x^2 = 3$, $\lim_{x \to 0} 3 + x^2 = 3$, by the Squeeze theorem, $\lim_{x \to 0} f(x) = 3$.

Example. Show that if $\lim_{x \to a} |f(x)| = 0$, then $\lim_{x \to a} f(x) = 0$.

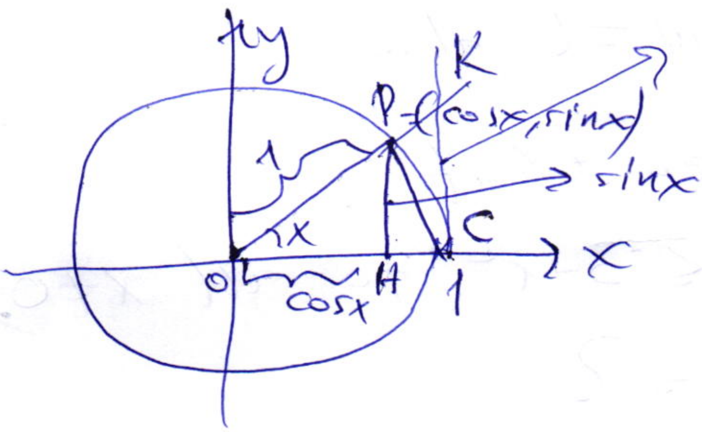
Solution: Since

$-|f(x)| \le f(x) \le |f(x)|$ and

$\lim_{x \to a} |f(x)| = 0$ and $\lim_{x \to a} -|f(x)| = 0$

$\Rightarrow \lim_{x \to a} f(x) = 0$ by Sq. th

Theorem. $\lim_{x \to 0} \frac{\sin x}{x} = 1$.



Let $0 < x < \pi/2$.

$\frac{\cos x \cdot \sin x}{2} = \text{Area}(\overset{\triangle}{OHP})$,

$\frac{\sin x}{2} = \text{Area}(\overset{\triangle}{OCP})$

$\frac{1 \cdot x}{2} = $ Area of sector (OCP)

$= \frac{1}{2} r^2 x$.

$\frac{\cos x \sin x}{2} < \frac{\sin x}{2} < \frac{x}{2}$

(5)

$|CK| = ?$ $\quad \dfrac{\sin x}{|CK|} = \dfrac{\cos x}{1} \rightarrow$

$$|CK| = \frac{\sin x}{\cos x} = \tan x.$$

$$\text{Area}(\overset{\triangle}{OCK}) = \frac{1}{2}\tan x.$$

sector

$$\text{Area}(\overset{\triangle}{OPC}) < \text{Area}(OCP) < \text{Area}(\overset{\triangle}{OCK})$$

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2} \quad 0 < x < \pi/2.$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\rightarrow \quad \cos x < \frac{\sin x}{x} < 1.$$

If $-\frac{\pi}{2} < x < 0$, then

$$\cos(-x) = \cos x, \quad \sin(-x) = -\sin x,$$

$$\cos x < \frac{\sin x}{x} < 1, \quad , \text{ if } 0 < |x| < \pi/2$$

$$\lim_{x\to 0} 1 = 1, \quad \lim_{x\to 0} \cos x = \cos 0 = 1.$$

by the Squeze th.,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Examples. Evaluate

① $\lim_{x \to -2} \frac{x^2+x-2}{x^2+5x+6} = ?$ $\left(\frac{0}{0}\right)$ indeter. form

$\lim_{\substack{x \to -2 \\ x \neq -2}} \frac{\cancel{(x+2)}(x-1)}{\cancel{(x+2)}(x+3)} = \lim_{x \to -2} \frac{x-1}{x+3}$

$= \frac{-2-1}{-2+3} = \frac{-3}{1} = -3.$

② $\lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x-4} = \frac{\frac{1}{4} - \frac{1}{4}}{4-4} = \left(\frac{0}{0}\right)$

indeterminate form,

$\lim_{x \to 4} \frac{\frac{4-x}{4x}}{x-4} = \cancel{\lim_{x \to 4} (4-x}$

$= \lim_{x \to 4} \frac{-(x-4)}{4x(x-4)} = \lim_{x \to 4} \frac{-1}{4x}$

$= -\frac{1}{16}$

③ $\lim_{x \to 4} \frac{\sqrt{x}-2}{x^2-16} = ?$ $\left(\frac{0}{0}\right)$ ind. f

$$\lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)}$$

$$= \lim_{\substack{x \to 4 \\ x \neq 4}} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(x+4)(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{1}{(x+4)(\sqrt{x}+2)}$$

$$= \frac{1}{(4+4)(2+2)} = \frac{1}{32}.$$

(4) Let $f(x) = \begin{cases} x, & x \neq 2 \\ 1, & x = 2 \end{cases}$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2.$$

(5) $\lim_{x \to 1} \frac{x^2-1}{x^2-2x+1} = \frac{1-1}{1-2+1} = \left(\frac{0}{0}\right)$ ind. f.

$\lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x-1)}$

(5) $\lim_{x \to -2} \frac{x^2+2x}{x^2-4} = \left(\frac{0}{0}\right)$

$\lim_{\substack{x \to -2 \\ x \neq -2}} \frac{\cancel{(x+2)}x}{\cancel{(x+2)}(x-2)} =$

$\lim_{x \to -2} \frac{x}{x-2} = \frac{-2}{-2-2} = \frac{-2}{-4} = \frac{1}{2}$

(6) $\lim_{x \to 0} \frac{\tan x}{x} = ?$

$\lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} =$

$= 1 \cdot 1 = 1$

(7) $\lim_{x \to 1} \frac{\sin(x^2-1)}{x-1} = ?$

$\lim_{x \to 1} \frac{\sin(x^2-1) \cdot (x+1)}{(x-1)(x+1)}$

$\lim_{x \to 1} \frac{\sin(x^2-1)}{x^2-1} \cdot \lim_{x \to 1} (x+1)$

Let $x^2-1 = u$. $\lim_{u \to 0} \frac{\sin u}{u} \cdot 2 = 1 \cdot 2 = 2.$