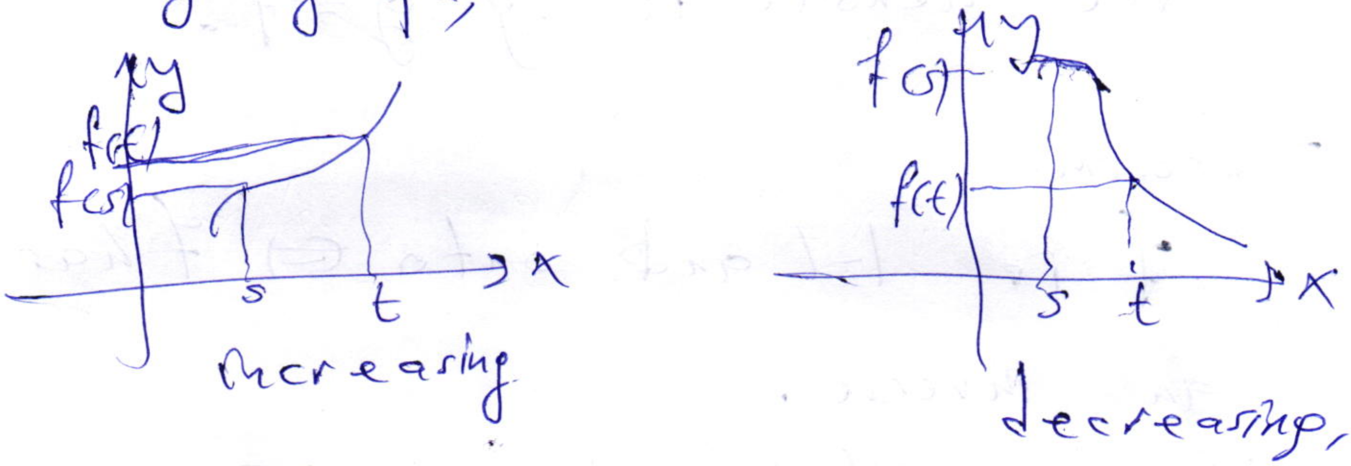
Definition. Let $f: X \to \mathbb{R}$ be a function.

f is called increasing on X if $f(s) < f(t)$ whenever $s < t$ in X.

f is called decreasing on X if $f(s) > f(t)$ whenever $s < t$ in X.

By graph,



Example. Show that $f(x) = x^2$ is increasing on $[0, \infty)$.

Solution. If $0 \le s < t < \infty$, then

$$f(t) - f(s) = t^2 - s^2 = \underbrace{(t+s)}_{>0}\underbrace{(t-s)}_{>0} > 0$$

Hence, f is increasing.

Definition. A function $I : X \to X$ is called identity if $I(x) = x$ for all $x \in X$.

Let $f: X \to Y$ and $g: Y \to X$ be functions.

If $g \circ f = I$, then ~~g~~ we say that g is left inverse of f.

If $f \circ g = I$, then we say that g is right inverse of f.

We denote it by $g = f^{-1}$.

Theorem.

f is 1-1 and onto $\iff$ f has an inverse.

Example. Let $f(x) = 3x + 2$.

Find $f^{-1}(x) = ?$

Solution: $f \circ f^{-1} = I \Rightarrow (f \circ f^{-1})(x) = I(x) = x$

$f(f^{-1}(x)) = x \Rightarrow$

$3 f^{-1}(x) + 2 = x \Rightarrow f^{-1}(x) = \dfrac{x-2}{3}$

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Trigonometric Functions

$$\sin\theta = \frac{|AB|}{|OA|}, \quad \cos\theta = \frac{|OB|}{|OA|}$$

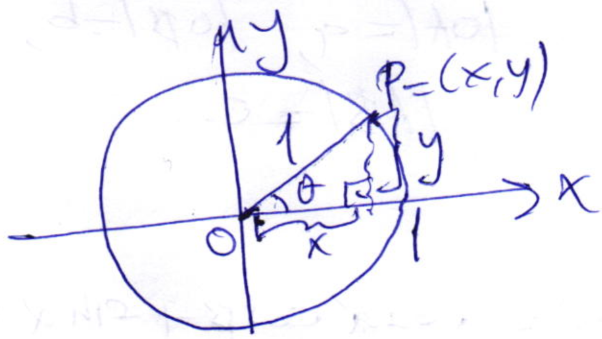
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{|AB|}{|OB|}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{|OB|}{|AB|}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{|OA|}{|OB|}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{|OA|}{|AB|}.$$

Let's think of $x^2+y^2=1$.



$$\cos\theta = \frac{x}{1} = x$$

$$\sin\theta = \frac{y}{1} = y.$$

So $P=(x,y) = (\cos\theta, \sin\theta)$.

Properties:

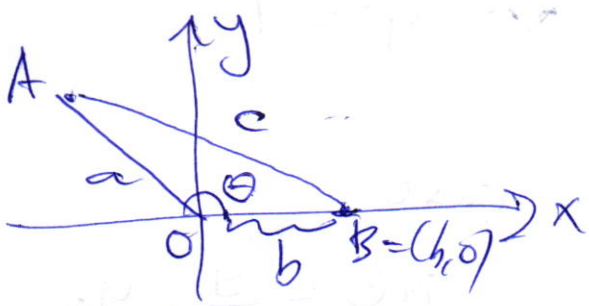
1) $\cos^2\theta + \sin^2\theta = 1$
2) $1+\tan^2\theta = \sec^2\theta$
3) $1+\cot^2\theta = \csc^2\theta$.

4) $|\cos\theta| \le 1$, $|\sin\theta| \le 1$.

5) $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$.

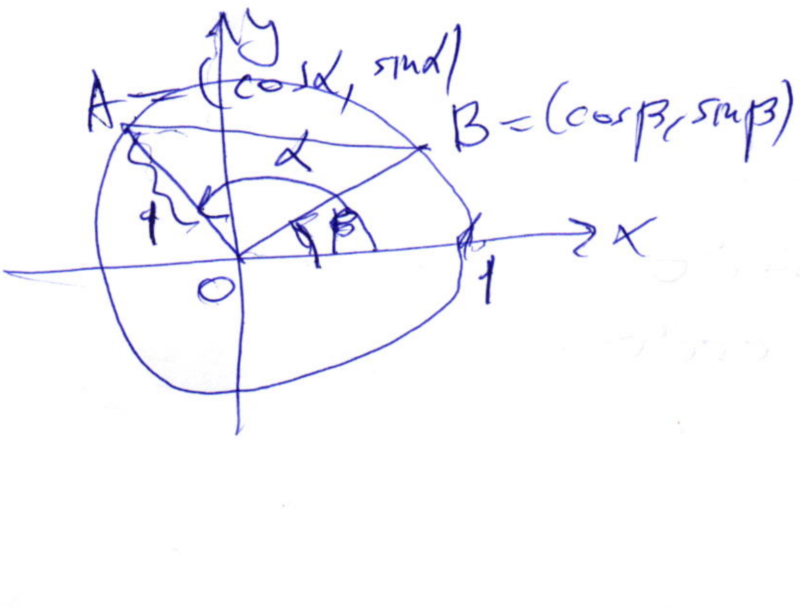
Law of cosines:
Let OAB be any triangle and let θ be the angle at the vertrces O.

Then,
$$c^2 = a^2 + b^2 - 2ab\cos\theta,.$$



$|OA| = a$, $|OB| = b$,
$|AB| = c$.

Theorem. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
<u>Proof.</u> Let $\alpha > \beta$.



By the cosine law,

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA| \cdot |OB| \cos(\alpha - \beta)$$

$$|AB|^2 = 1 + 1 - 2\cos(\alpha - \beta)$$

Also,

$$|AB|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$$= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$$

Therefore,

$$\cancel{2} - \cancel{2}\cos(\alpha - \beta) = \cancel{2} - \cancel{2}[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$$

$$\Rightarrow \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Identities:

1) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
2) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
4) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$
5) $\sin 2\theta = 2\sin\theta\cos\theta$
6) $\cos 2\theta = \cos^2\theta - \sin^2\theta.$
7) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
8) $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

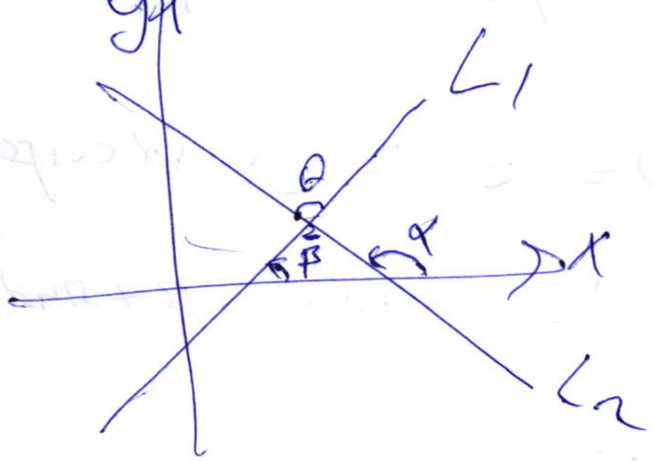
(6)

9) $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

the angle between two lines

$\tan\theta = \dfrac{m_2 - m_1}{1+m_1 m_2}$

$\tan\alpha = m_2, \quad \tan\beta = m_1$



$\theta + \beta = \alpha \Rightarrow \theta = \alpha - \beta.$

Limit

Definition. Let $a \in \mathbb{R}$, $\delta > 0$.

1) δ-neighborhood of a = $\{x \mid |x-a| < \delta\} = (a-\delta, a+\delta)$

2) δ-deleted neighborhood of a $= \{x \mid 0 < |x-a| < \delta\} = (a-\delta, a) \cup (a, a+\delta)$

Definition. Let f be defined in a deleted neighborhood of a.

$\lim_{x \to a} f(x) = L \in \mathbb{R} \iff$ given any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $0 < |x - a| < \delta$.

Example. Show that

$\lim_{x \to 1} (3x+2) = 5$.

Solution: Given any $\varepsilon > 0$ find $\delta > 0$? such that

$$|(3x+2) - 5| = |3(x-1)| = 3|x-1| \overset{?}{<} \varepsilon$$

whenever $0 < |x-2| < \delta$.

Take $\delta = \frac{\varepsilon}{3}$. $\Rightarrow$

$$|(3x+2) - 5| < \varepsilon.$$

Example. Show that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Solution. Given any $\varepsilon > 0$, find $\delta > 0$? such that

$$\left|\frac{x^2-1}{x-1} - 2\right| = \left|\frac{(x-1)(x+1)}{x-1} - 2\right|$$

$$= |x+1-2| = |x-1| < \varepsilon$$

whenever $0 < |x-1| < \delta$.

Choose $\delta = \varepsilon$.