1/a. Find and classify the critical points of $f(x,y) = x^2 + 2y^2 - 4x + 4y$

Solution.

$f_x = 2x - 4 = 0 \longrightarrow x = 2$

$f_y = 4y + 4 = 0 \longrightarrow y = -1.$

Critical point is $P = (2,-1)$.

$f_{xx} = 2, \quad f_{xy} = 0 = f_{yx}$

$f_{yy} = 4.$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 > 0.$$

$f_{xx}(P) = 2 > 0 \Rightarrow f$ has a local min. at P.

Minimum value is $f(2,-1) = 4 + 2 + 8 - 4$
$= 10.$

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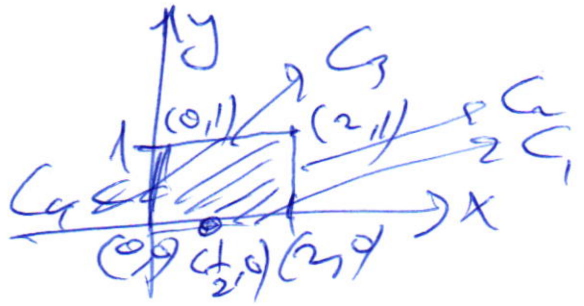
2/ Find the maximum and minimum values of $f(x,y) = x - x^2 + y^2$ on rectangle $0 \le x \le 2$, $0 \le y \le 1$.



$f_x = 1 - 2x = 0 \rightarrow x = 1/2$

$f_y = 2y = 0 \rightarrow y = 0$.

Crt.pt. $P_1 = (\frac{1}{2}, 0)$.

on C_1: $0 \le x \le 2$, $y = 0$.

$g(x) = f(x,0) = x - x^2$

$g'(x) = 1 - 2x = 0 \rightarrow x = 1/2 \Rightarrow y = 0$.

$(\frac{1}{2}, 0)$

End points: $(0,0)$, $(2,0)$. crt.pts.

on C_2: $x = 2$, $0 \le y \le 1$.

$g(y) = f(2,y) = 2 - 4 + y^2 = -2 + y^2$.

$g'(y) = 2y = 0 \rightarrow y = 0 \Rightarrow P = (2,0)$ ✓

End points: $(2,0)$, $(2,1)$. crt.pts

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On C_3: $0 \le x \le 2$, $y = 1$.

$g(x) = f(x,1) = x - x^2 + 1$

$g'(x) = 1 - 2x = 0 \rightarrow x = 1/2$

$P(\frac{1}{2}, 1)$.

End points: $(0,1)$, $(2,1)$.

On C_4: $x = 0$, $0 \le y \le 1$.

$g(y) = f(0,y) = y^2 \rightarrow g'(y) = 2y = 0 \rightarrow y = 0$.

$\Rightarrow P = (0,0)$. ✓

End points: $(0,0)$, $(0,1)$.

All critical points:

$P_1 = (\frac{1}{2}, 0)$, $P_2 = (0,0)$, $P_3 = (2,0)$,

$P_4 = (2,1)$, $P_5 = (0,1)$, $P_6 = (\frac{1}{2}, 1)$.

$f(P_1) = f(\frac{1}{2}, 0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

$f(P_2) = f(0,0) = 0$.

$f(P_3) = f(2,0) = 2 - 4 = -2$.

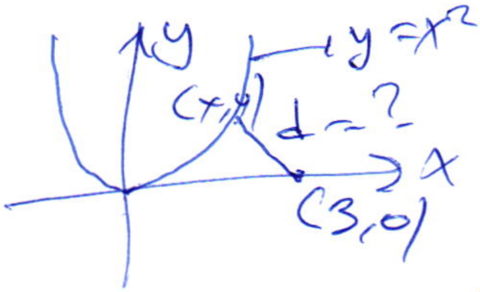
$f(P_4) = f(2,1) = 2 - 4 + 1 = -1$

$f(P_5) = f(0,1) = 1$

$f(P_6) = f(\frac{1}{2}, 1) = \frac{1}{2} - \frac{1}{4} + 1 = \frac{2 - 1 + 4}{4} = \frac{5}{6}$.

Minimum is -2, Maximum is 1

5) Find the shortest distance from $(3,0)$ to $y=x^2$.



$$d^2 = (x-3)^2 + y^2 = f(x,y).$$

constraint: $g(x,y) = y - x^2 = 0.$

Lagrange function

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$L(x,y,\lambda) = (x-3)^2 + y^2 + \lambda(y-x^2).$$

$$0 = \frac{\partial L}{\partial x} = 2(x-3) - 2x\lambda \rightarrow x-3-x\lambda=0$$

$$x(1-\lambda) = 3$$

$$x = 3/(1-\lambda)$$

$$0 = \frac{\partial L}{\partial y} = 2y + \lambda = 0 \rightarrow y = -\lambda/2$$

$$0 = \frac{\partial L}{\partial \lambda} = y - x^2.$$

$$y - x^2 = 0 \rightarrow -\frac{\lambda}{2} - \left(\frac{3}{1-\lambda}\right)^2 = 0.$$

$$-\frac{\lambda}{2} - \frac{9}{(1-\lambda)^2} = 0$$

$$-\frac{\lambda}{2} = \frac{9}{(1-\lambda)^2} \Rightarrow \lambda(1-\lambda)^2 = -18.$$

$$\Rightarrow \lambda(1-2\lambda+\lambda^2) + 18 = 0$$

$$\lambda - 2\lambda^2 + \lambda^3 + 18 = 0$$

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5/ Find the distance from the origin to the plane $x+2y+2z=3$.

Solution.

Distance function

$d^2 = f(x,y,z) = x^2+y^2+z^2$.

Constraint, $g(x,y,z) = x+2y+2z-3=0$.

Lagrange function =

$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$.

$= x^2+y^2+z^2+\lambda(x+2y+2z-3)$.

$0 = \frac{\partial L}{\partial x} = 2x+\lambda \longrightarrow x = -\frac{\lambda}{2}$

$0 = \frac{\partial L}{\partial y} = 2y+2\lambda \longrightarrow y = -\lambda$

$0 = \frac{\partial L}{\partial z} = 2z+2\lambda \longrightarrow z = -\lambda$

$0 = \frac{\partial L}{\partial \lambda} = x+2y+2z-3$

$0 = x+2y+2z-3 = \left(-\frac{\lambda}{2}\right)+2(-\lambda)+2(-\lambda)-3$

$-\frac{\lambda}{2} - 2\lambda - 2\lambda - 3 = 0 \rightarrow -\frac{\lambda}{2} - 4\lambda = 3$

$-\frac{\cancel{9}\lambda}{3} = 2.\cancel{3} \rightarrow \lambda = -2/3$, $\quad \frac{(-1-8)\lambda}{2} = 3$

So, $x = -\frac{1}{2}(-2/3) = 1/3$, $y = 2/3$, $z = 2/3$.

$d^2 = f(1/3, 2/3, 2/3) = \frac{1}{9} + \frac{4}{9} + 4/9 = \frac{9}{9} = 1$.

$\underline{d = 1}$ dist.

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$\lambda = 2 \rightarrow 2 - 8 + 8 + 18 \neq 0$

$\lambda = 3 \rightarrow 3 - 2 \cdot 9 + 27 + 18$

$\lambda = 6 \rightarrow 6 - 2 \cdot 36 + 36 \cdot 6 + 18$