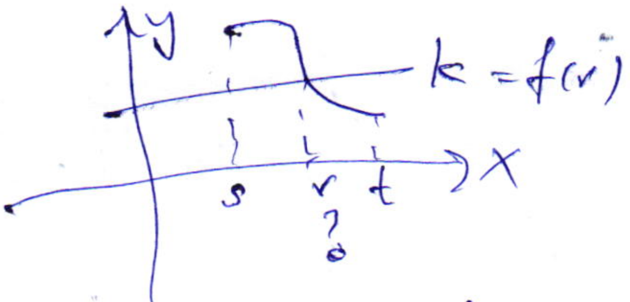
Intermediate value theorem:
If f is continuous on an interval I and if f takes different values $f(s)$ and $f(t)$ at two points s and t of I, then f takes every intermediate value k, that is, every value k between $f(s)$ and $f(t)$ at some point r between s and t.



Example. Show that the equation

$2x^5+2x^2+x-3=0$ has a root

between 0 and 1.

Solution: $f(x)=2x^5+2x^2+x-3$ is a continuous function.

$f(0)=-3<0$, $f(1)=2>0$

$-3<0<2$, $k=0$. There is a root $r\in(0,1)$ such that

$f(r)=0$.

Extreme Value theorem:
If f is continuous on a bounded closed interval $I=[a,b]$, then f is bounded on I and f has both a maximum M and a minimum m on I, that is, there are points p and q in I such that

$$m=f(p)\leq f(x)\leq f(q)=M$$

for all $x\in I$.

Example. Let $f(x)=\frac{1}{\sqrt{x}}$ on $I=[1,4]$.

$$f(1)=1=\underbrace{M}_{\text{maximum}}, \quad f(4)=\frac{1}{2}=\underbrace{m}_{\text{minimum}}.$$

Derivatives

Definition. Let f be a function defined in a neighborhood of a point x. Then, by the derivative of at x, denoted by $f'(x)$, we mean the limit

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

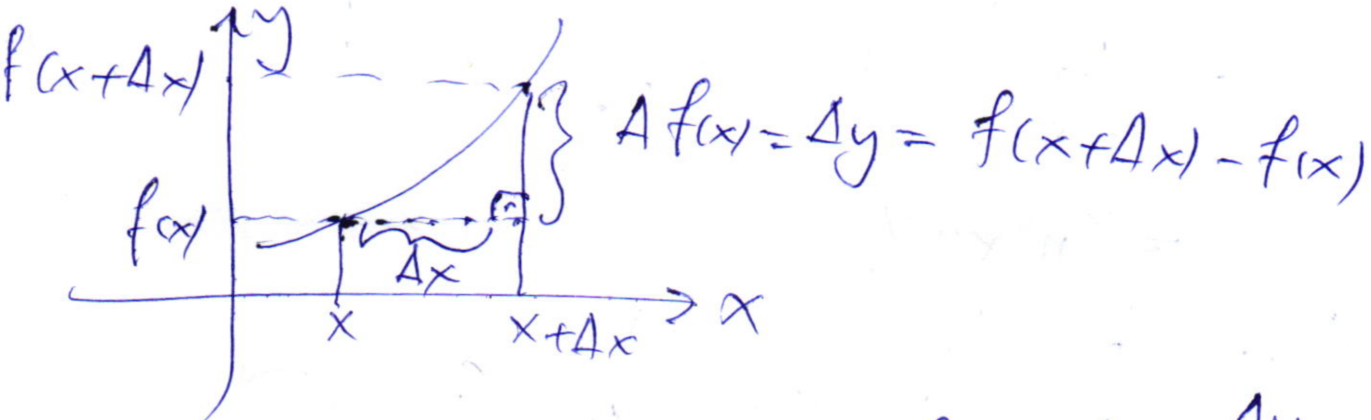
provided that the limit exists.

Equivalently,

$$\lim_{u \to s} \frac{f(u) - f(s)}{u - s} = f'(s), \quad \text{Here,}$$

$u = x + \Delta x$.

If f has a derivative at x, we say that f is differentiable at x.



$$y = f(x) \Rightarrow y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = D f(x)$$

Example. Let $f(x) = c$, $c \in \mathbb{R}$.

Find $f'(x) = ?$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0.$$

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Example. Let $f(x)=x^n$, $n\in\mathbb{N}$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}(\Delta x)^2+\dots+(\Delta x)^n-x^n}{\Delta x}$$

$$=\lim_{\Delta x\to 0} nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}\Delta x+\dots+(\Delta x)^{n-1}$$

$$=nx^{n-1}.$$

Example. Let $f(x)=\sqrt{x}$.

$$\frac{d}{dx}f(x)=\lim_{u\to x}\frac{\sqrt{u}-\sqrt{x}}{u-x}=\lim_{u\to x}\frac{(\sqrt{u}-\sqrt{x})}{(\sqrt{u}-\sqrt{x})(\sqrt{u}+\sqrt{x})}$$

$$=\lim_{u\to x}\frac{1}{\sqrt{u}+\sqrt{x}}=\frac{1}{2\sqrt{x}},\ x>0$$

Example Let $f(x)=\sin x$.

$$\frac{d}{dx}\sin x=\lim_{u\to x}\frac{\sin u-\sin x}{u-x}=\lim_{u\to x}\frac{2\cos\left(\frac{u+x}{2}\right)\sin\left(\frac{u-x}{2}\right)}{u-x}$$

$$=\lim_{u\to x}\cos\left(\frac{u+x}{2}\right)\lim_{u\to x}\frac{\sin\left(\frac{u-x}{2}\right)}{\frac{u-x}{2}}$$

$$=\cos x\cdot 1=\cos x$$

$$\Downarrow$$

$$\frac{u-x}{2}=t$$

$$\lim_{t\to 0}\frac{\sin t}{t}=1$$

Theorem. If f and g are differentiable at x, then so is the sum $f+g$, and

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Theorem. If the functions f and g are differentiable at x, then so is the product $f \cdot g$ and

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = g(x)\frac{df(x)}{dx} + f(x) \cdot \frac{dg(x)}{dx}$$

Theorem. If the functions f and g are differentiable at x, then so is the quotient $\frac{f(x)}{g(x)}$ and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

provided that $g(x) \neq 0$.

Theorem. If f is a differentiable function at $x=a$, then f is continuous at $x=a$.

Proof. We must show $\lim_{x \to a} f(x) = f(a)$.

or equivalently, $\lim_{x \to a} (f(x) - f(a)) = 0$

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot (x-a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot \lim_{x \to a} (x-a)$$

$$= f'(a) \cdot 0 = 0.$$

Therefore, f is continuous at $x=a$.