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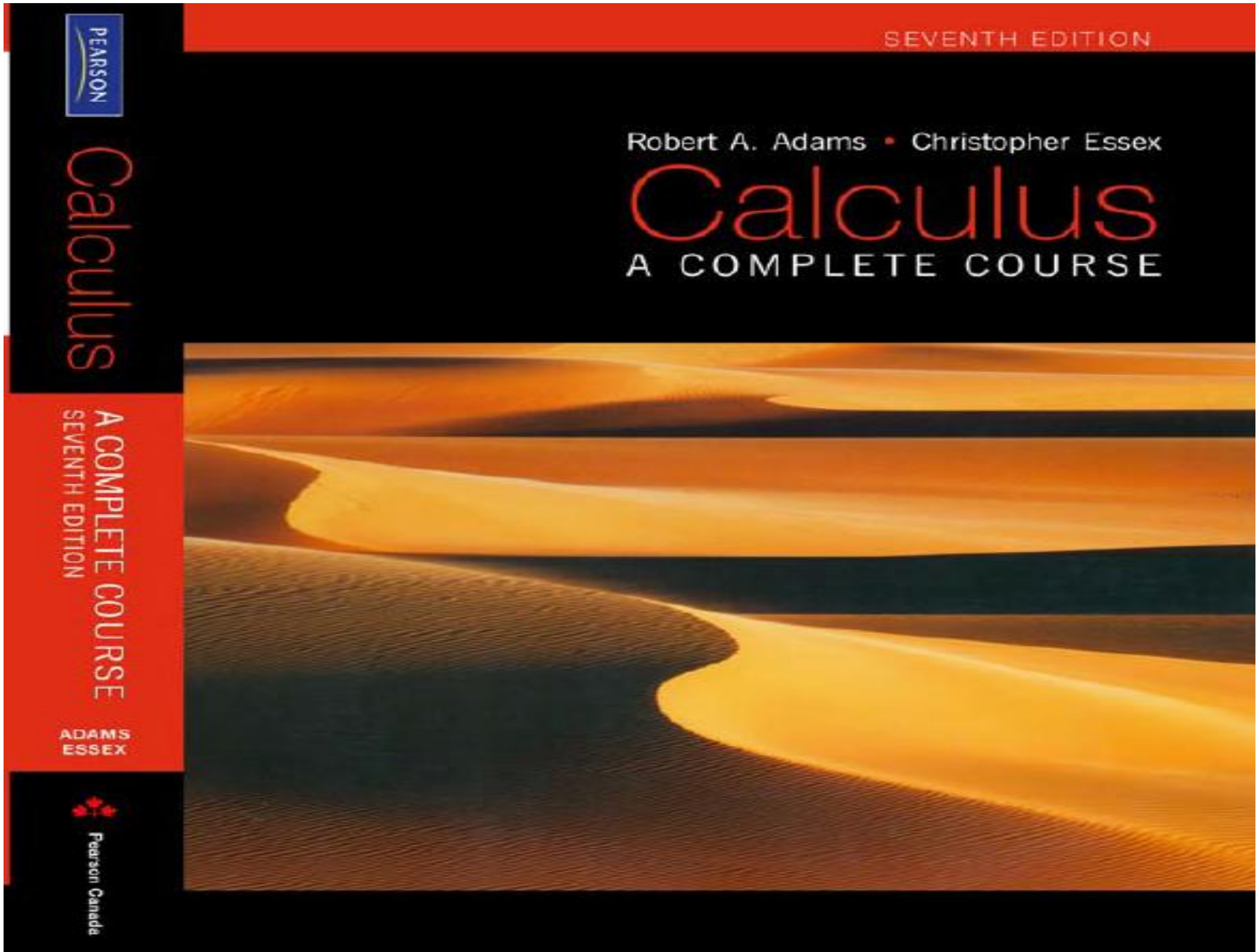
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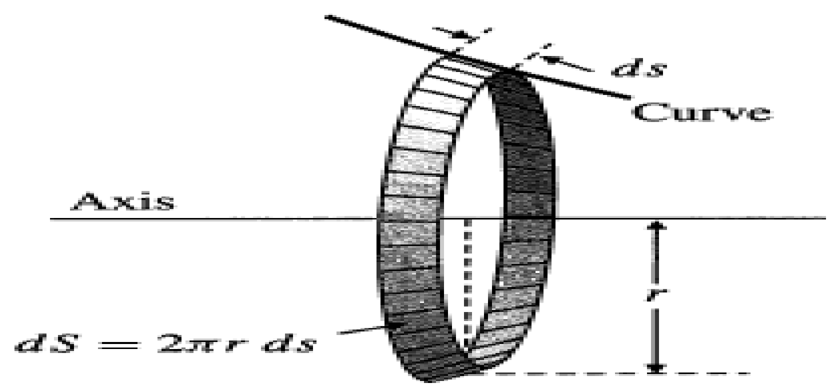
Areas of Surfaces of Revolution

When a plane curve is rotated (in three dimensions) about a line in the plane of the curve, it sweeps out a **surface of revolution**. For instance, a sphere of radius a is generated by rotating a semicircle of radius a about the diameter of that semicircle.

The area of a surface of revolution can be found by integrating an area element dS constructed by rotating the arc length element ds of the curve about the given line. If the radius of rotation of the element ds is r, then it generates, on rotation, a circular band of width ds and length (circumference) $2\pi r$. The area of this band is, therefore,

$$dS = 2\pi r\, ds,$$

as shown in Figure The areas of surfaces of revolution around various lines can be obtained by integrating dS with appropriate choices of r. Here are some important special cases.



The circular band generated by rotating arc length element ds about the axis

Area of a surface of revolution

If $f'(x)$ is continuous on $[a, b]$ and the curve $y = f(x)$ is rotated about the x-axis, the area of the surface of revolution so generated is

$$S = 2\pi \int_{x=a}^{x=b} |y|\, ds = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx.$$

If the rotation is about the y-axis, the surface area is

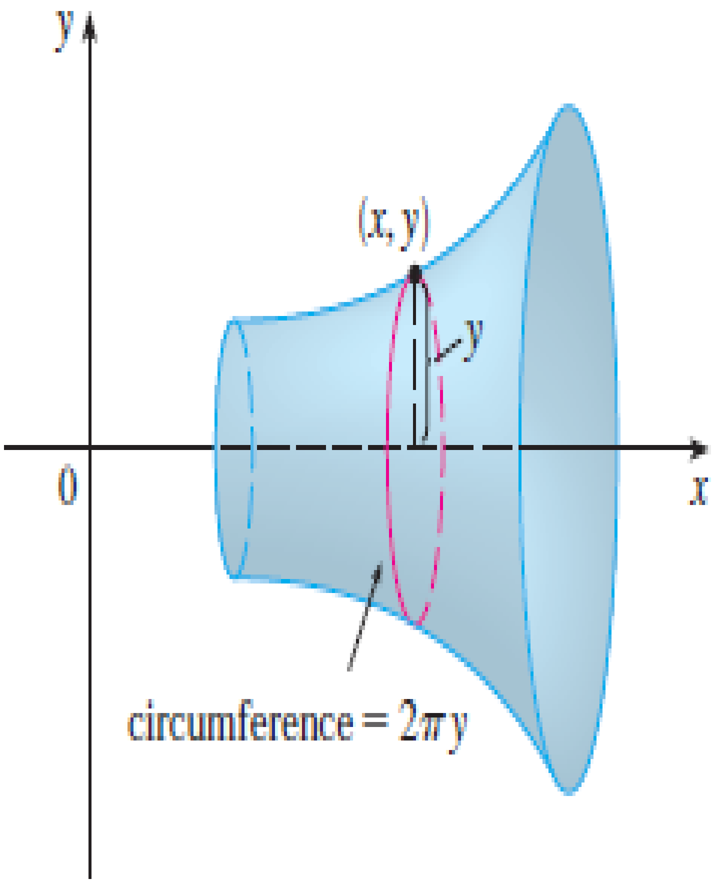
$$S = 2\pi \int_{x=a}^{x=b} |x|\, ds = 2\pi \int_a^b |x|\sqrt{1 + (f'(x))^2}\, dx.$$

If $g'(y)$ is continuous on $[c, d]$ and the curve $x = g(y)$ is rotated about the x-axis, the area of the surface of revolution so generated is

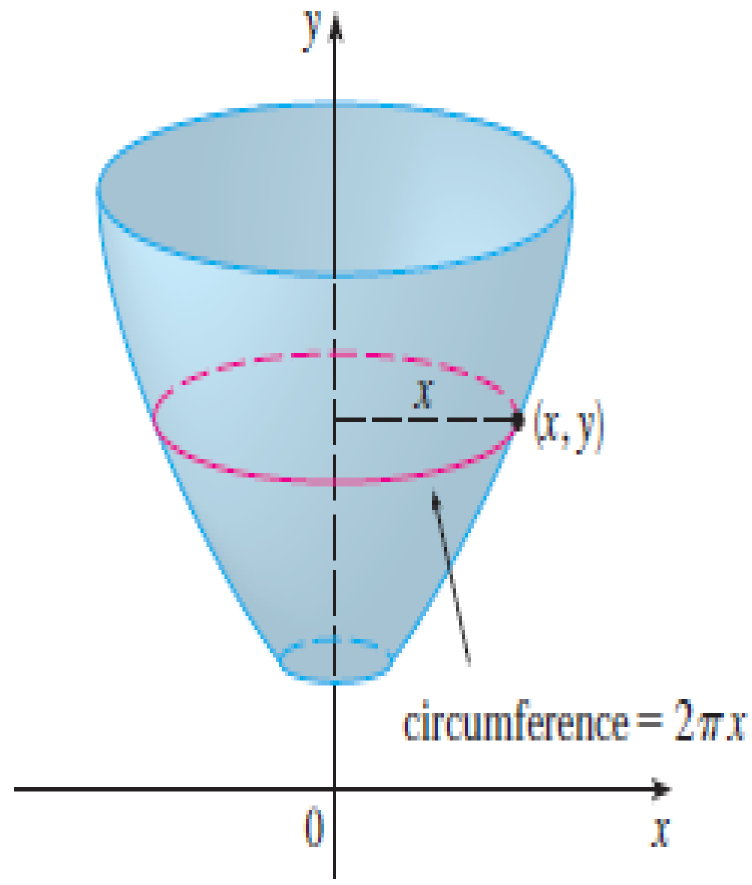
$$S = 2\pi \int_{y=c}^{y=d} |y|\, ds = 2\pi \int_c^d |y|\sqrt{1 + (g'(y))^2}\, dy.$$

If the rotation is about the y-axis, the surface area is

$$S = 2\pi \int_{y=c}^{y=d} |x|\, ds = 2\pi \int_c^d |g(y)|\sqrt{1 + (g'(y))^2}\, dy.$$



(a) Rotation about x-axis: $S = \int 2\pi y \, ds$



(b) Rotation about y-axis: $S = \int 2\pi x \, ds$

(Surface area of a sphere) Find the area of the surface of a sphere of radius a.

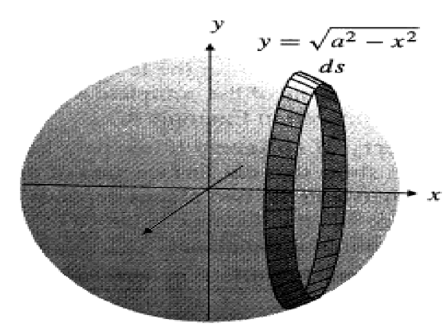
Solution Such a sphere can be generated by rotating the semicircle with equation

$y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the x-axis. Since

$$\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y},$$

the area of the sphere is given by

$$\begin{aligned} S &= 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{x}{y}\right)^2}\, dx \\ &= 4\pi \int_0^a \sqrt{y^2 + x^2}\, dx \\ &= 4\pi \int_0^a \sqrt{a^2}\, dx = 4\pi a x \Big|_0^a = 4\pi a^2 \text{ square units.} \end{aligned}$$

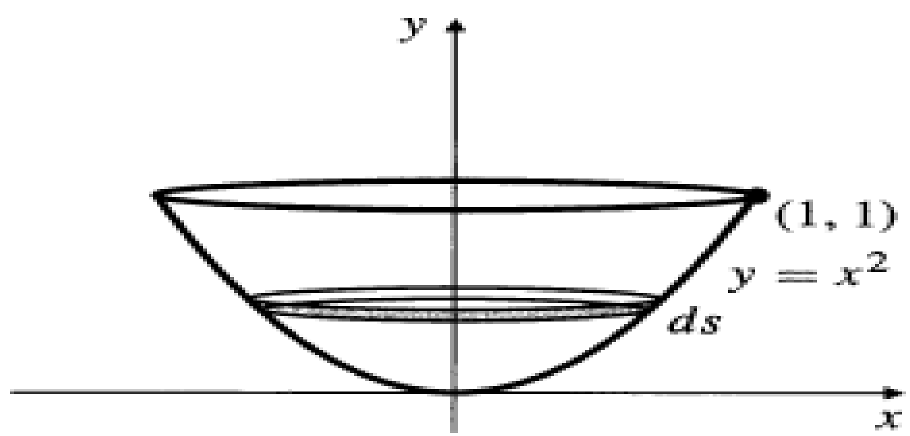


An area element on a sphere

(Surface area of a parabolic dish) Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $(0 \le x \le 1)$, about the y-axis, as illustrated in Figure

Solution The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1+4x^2}\,dx$, so the required surface area is

$$\begin{aligned} S &= 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx && \text{Let } u = 1+4x^2, \\ & && du = 8x\,dx \\ &= \frac{\pi}{4}\int_1^5 u^{1/2}\,du \\ &= \frac{\pi}{6}\,u^{3/2}\Big|_1^5 = \frac{\pi}{6}\,(5\sqrt{5}-1) \text{ square units.} \end{aligned}$$



The area element is a horizontal band here

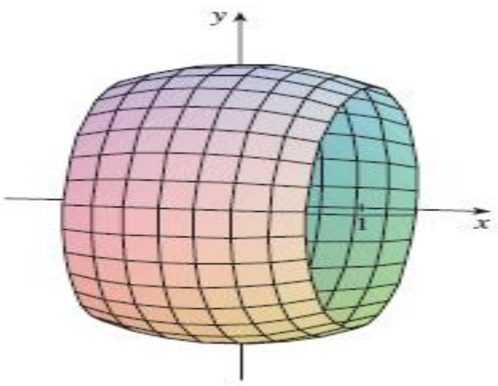
EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is sphere of radius 2. See Figure)

SOLUTION We have

$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$\begin{aligned} S &= \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \frac{2}{\sqrt{4 - x^2}}\, dx \\ &= 4\pi \int_{-1}^{1} 1\, dx = 4\pi(2) = 8\pi \end{aligned}$$



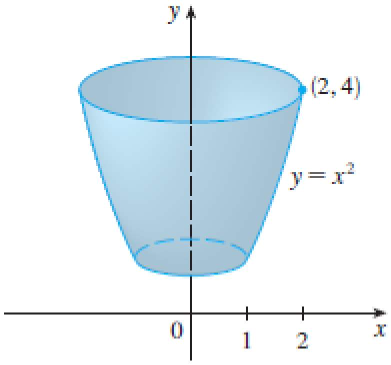
FIGURE

The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

SOLUTION Using

$$y = x^2 \quad \text{and} \quad \frac{dy}{dx} = 2x$$

we have, from Formula 8,



$$\begin{aligned} S &= \int 2\pi x \, ds \\ &= \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\ &= 2\pi \int_1^2 x \sqrt{1 + 4x^2} \, dx \end{aligned}$$

Substituting $u = 1 + 4x^2$, we have $du = 8x\,dx$. Remembering to change the limits of integration, we have

$$\begin{aligned} S &= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du = \frac{\pi}{4} \left[\tfrac{2}{3} u^{3/2}\right]_5^{17} \\ &= \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

SOLUTION 2 Using

$$x = \sqrt{y} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

we have

$$\begin{aligned} S &= \int 2\pi x\, ds = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy = \pi \int_1^4 \sqrt{4y + 1}\, dy \\ &= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du \qquad \text{(where } u = 1 + 4y\text{)} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \qquad \text{(as in Solution 1)} \end{aligned}$$

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis.

SOLUTION

$$y = e^x \qquad \text{and} \qquad \frac{dy}{dx} = e^x$$

we have

$$\begin{aligned}
S &= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx \\
&= 2\pi \int_1^e \sqrt{1 + u^2}\, du \qquad \text{(where } u = e^x\text{)} \\
&= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta \qquad \text{(where } u = \tan\theta \text{ and } \alpha = \tan^{-1}e\text{)} \\
&= 2\pi \cdot \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\alpha} \\
&= \pi\Big[\sec\alpha\tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln\big(\sqrt{2} + 1\big)\Big]
\end{aligned}$$

Since $\tan\alpha = e$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\Big[e\sqrt{1 + e^2} + \ln\big(e + \sqrt{1 + e^2}\big) - \sqrt{2} - \ln\big(\sqrt{2} + 1\big)\Big]$$

HOMEWORK

The following parts involve the curve

$y = x^2 \sin(2x)$ from $x=0$ to $x=\frac{\pi}{2}$

a) write, but do NOT evaluate, the integral which gives the arclength of $y = x^2 \cdot \sin(2x)$ from $x=0$ to $x=\frac{\pi}{2}$

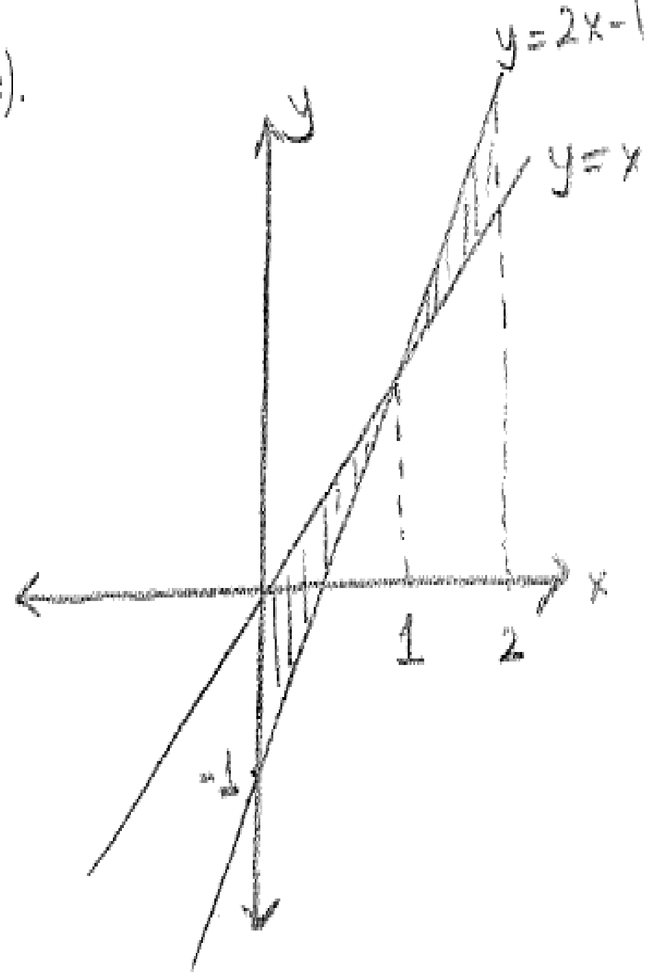
b) write, but do NOT evaluate, the integral which gives the surface area of the surface obtained by rotating this curve about X-axis

c) write, but do NOT evaluate, the integral which gives the surface area of the surface obtained by rotating th curve about $x=-2$ line

SOME EXAMPLES:

In the following parts, R is the region between $y = x$ and $y = 2x - 1$ from $x = 0$ to $x = 2$.

(a) Compute the area of the region R (described above).



$$\text{Area} = \int_0^2 \left| (2x-1) - x \right| dx$$

$$= \int_0^1 1-x \, dx + \int_1^2 x-1 \, dx$$

$$= x - \frac{x^2}{2} \Big|_0^1 + \frac{x^2}{2} - x \Big|_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left(0 - \left(-\frac{1}{2}\right)\right) = 1$$

DİSK METHOD

(b) Compute the volume of R rotated around $y = -1$.

$$\text{Volume} = \int_0^1 \pi\left[(x-(-1))^2 - ((2x-1)-(-1))^2\right]dx + \int_1^2 \pi\left[((2x-1)-(-1))^2 - (x-(-1))^2\right]dx$$

$$= \pi\int_0^1 \left[(x+1)^2 - (2x)^2\right]dx + \pi\int_1^2 \left[(2x)^2 - (x+1)^2\right]dx$$

$$= \pi\int_0^1 (x^2+2x+1-4x^2)\,dx + \pi\int_1^2 (4x^2 - x^2 - 2x - 1)\,dx$$

$$= \pi\left(-x^3+x^2+x\right)\Big|_0^1 + \pi\left(x^3-x^2-x\right)\Big|_1^2$$

$$= \pi(1-0) + \pi(2-(-1)) = 4\pi$$

SHELL METHOD

(c) Compute the volume of R rotated around $x = -1$.

$$\text{Volume} = \int_0^1 2\pi\,(x-(-1))\,(x-(2x-1))\,dx + \int_1^2 2\pi\,(x-(-1))\,((2x-1)-x)\,dx$$

$$= \int_0^1 2\pi\,(x+1)(1-x)\,dx + \int_1^2 2\pi\,(x+1)(x-1)\,dx$$

$$= 2\pi \int_0^1 (1-x^2)\,dx + 2\pi \int_1^2 (x^2-1)\,dx$$

$$= 2\pi \left(x - \frac{x^3}{3}\right)\Big|_0^1 + 2\pi \left(\frac{x^3}{3} - x\right)\Big|_1^2$$

$$= 2\pi \left(\frac{2}{3} - 0\right) + 2\pi \left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right)$$

$$= \frac{4\pi}{3} + \frac{8\pi}{3} = 4\pi$$

$$\int_2^\infty \frac{3-\arctan x}{x^{2/3}-1}\,dx = ? \qquad \text{(Use comparison test)}$$

On $(1,\infty)$ $\quad 0 < \arctan x < \frac{\pi}{2} < 2$

$$x^{2/3}-1 < x^{2/3}$$

So, $$\frac{3-\arctan x}{x^{2/3}-1} > \frac{1}{x^{2/3}}$$

$$\underbrace{\int_2^\infty \frac{3-\arctan x}{x^{2/3}-1}\,dx}_{\text{divergent by comparison}} > \underbrace{\int_2^\infty \frac{1}{x^{2/3}}\,dx}_{\text{divergent by p-test}}$$

$$\int_{-1}^{1} \frac{1}{x^2+2x}\,dx = \int_{-1}^{1} \frac{1}{x(x+2)}\,dx = \int_{-1}^{0} \frac{1}{x(x+2)}\,dx + \int_{0}^{1} \frac{1}{x(x+2)}\,dx$$

$$\int_{0}^{1} \frac{1}{x(x+2)}\,dx = \underbrace{\frac{1}{2}\int_{0}^{1} \frac{1}{x}\,dx}_{\text{Divergent by p-test}} - \frac{1}{2}\int_{0}^{1} \frac{1}{x+2}\,dx$$

Hence, $\int_{-1}^{1} \frac{1}{x(x+2)}\,dx$ is divergent

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}.$$

(a) Find the radius of convergence.

$$\sum_{n=1}^{\infty} \left| \frac{n^2 x^n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{n^2 |x|^n}{2^n}$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2|x|^{n+1}}{2^{n+1}}}{\frac{n^2|x|^n}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2|x|^{n+1}2^n}{n^2|x|^n 2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2|x|^n|x|2^n}{n^2|x|^n 2^n 2} = \lim_{n\to\infty} \frac{(n+1)^2|x|}{2n^2} = \frac{|x|}{2}$$

$$\frac{|x|}{2} < 1 \Leftrightarrow |x| < 2 \Rightarrow \text{ radius of convergence } = 2$$

- Taking the absolute value correctly and obtaining $\sum_{n=1}^{\infty} \frac{n^2|x|^n}{2^n}$:
- For the ratio test, setting up the limit $\lim_{n\to\infty} \frac{\frac{(n+1)^2|x|^{n+1}}{2^{n+1}}}{\frac{n^2|x|^n}{2^n}}$:
- Evaluating the limit and getting $\frac{|x|}{2}$:
- From the limit, obtaining the radius of convergence 2:

(b) Determine the convergence at endpoints.

$$x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n} = \sum_{n=1}^{\infty} \frac{n^2 2^n}{2^n} = \sum_{n=1}^{\infty} n^2, \ \lim_{n\to\infty} n^2 = \infty \neq 0 \Rightarrow \sum_{n=1}^{\infty} n^2 = \infty$$

$$x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n} = \sum_{n=1}^{\infty} \frac{n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$

$\lim_{n\to\infty} (-1)^n n^2$ does not exist $\Rightarrow \sum_{n=1}^{\infty} (-1)^n n^2$ diverges

- Finding $\sum_{n=1}^{\infty} n^2$ and stating the divergence of it with appropriate reason:
- Finding $\sum_{n=1}^{\infty} (-1)^n n^2$ and stating the divergence of it with appropriate reason:

(c) Write down the interval of convergence.

$(-2, 2)$

(a)
$$\sum_{n=1}^{\infty} \cos\left(\frac{2n-1}{n+1}\pi\right).$$

Solution: Since the function $y = \cos x$ is continuous on $\mathbb{R}$, we know that

$$\lim_{n\to\infty} \cos\left(\frac{2n-1}{n+1}\pi\right) = \cos\left(\lim_{n\to\infty} \frac{2n-1}{n+1}\pi\right) = \cos 2\pi = 1 \neq 0.$$

By n-th term test, the series is divergent.

(b)
$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}.$$

Solution: Let $a_n = \left(1 + \frac{1}{n}\right)^{-n^2}$ for $n \geq 1$. Then $a_n > 0$ for all n and $\sqrt[n]{a_n} = \left(1 + \frac{1}{n}\right)^{-n}$. Hence

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

By root test, we find that the series is convergent.

(c) $$\sum_{n=1}^{\infty} \frac{(n+4)!}{3!n!3^n}.$$

Solution: Let $a_n = \dfrac{(n+3)!}{3!n!3^n}$ for $n \geq 1$. Then $a_n > 0$ for all n and

$$\frac{a_{n+1}}{a_n} = \frac{(n+4)!}{3!(n+1)!3^{n+1}} \cdot \frac{3!n!3^n}{(n+3)!} = \frac{n+4}{3(n+1)}.$$

Hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1.$$

By ratio test, the series is convergent.

(d) $$\sum_{n=3}^{\infty} \frac{5n^4 - 3n^3 + 2}{n^2(n+1)(n^2+5)}.$$

Solution: Let $a_n = \dfrac{5n^4 - 3n^3 + 2}{n^2(n+1)(n^2+5)}$ and $b_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 5 > 0.$$

By p-test, we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence by limit comparison test, the series is divergent.

Is the following infinite series absolutely convergent, or conditionally convergent, or divergent? Please give full explanation to your answer.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\sin\frac{1}{n}.$$

Solution: We know that $0 < 1/n < \pi/2$ for all $n \geq 1$. Hence $\sin\dfrac{1}{n} > 0$ for all $n \geq 1$. We see that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\sin\frac{1}{n}$ is an alternating series. Denote $b_n = \sin\dfrac{1}{n}$ for $n \geq 1$. Since the function $y = \sin x$ is increasing on $[0, \pi/2]$, and for $n \geq 1$, $\dfrac{1}{n+1} < \dfrac{1}{n}$, we see that

$$b_{n+1} = \sin\frac{1}{n+1} < \sin\frac{1}{n} = b_n, \quad n \geq 1.$$

Hence (b_n) is a decreasing sequence of real numbers. Since $y = \sin x$ is a continuous function on $\mathbb{R}$,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sin\frac{1}{n} = \qquad 0.$$

we find the infinite series is convergent. Let $a_n = 1/n$ for $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

By p-test, we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. By the limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$ is divergent. We conclude that the series is conditionally convergent.

Does the series $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$ converge or not? Indicate a reason, or which test is used and how.

Use the Ratio Test:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} \\
&= \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^n}{10^{n+1}} \\
&= \frac{1}{10} \cdot \left(1 + \frac{1}{n}\right)^{10} \\
&\xrightarrow[n\to\infty]{} \frac{1}{10} \cdot (1+0)^{10} \\
&= \frac{1}{10} < 1.
\end{aligned}$$

Hence, the series converges.

Does the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ converge? If yes, **compute** the series; if not, indicate a reason.

Using partial fractions, we see that

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

This allows us to compute explicitly the partial sums of this telescoping series:

$$\begin{aligned}
s_k &= \sum_{n=1}^{k} \frac{1}{(2n-1)(2n+1)} \\
&= \sum_{n=1}^{k} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2k+1}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2k+1}\right)
\end{aligned}$$

Hence, $\lim_{k\to\infty} s_k = \frac{1}{2}$, and the series converges. The sum of the series, then, is the limit of the partial sums,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$