

Course Code	BME3310
Course Title	System Identification
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Course Assistant	Ümmühan Zengin Özer
Textbook	<a href="#">Ljung</a> System Identification: Theory for the User (1987) G. <a href="#">Platt</a> System ID Course Notes and Lecture Recordings
Weekly Assignments	GL Prett, <b>Lecture Recordings</b> (Indicated as “Movie1,2, etc.”) GL Prett, <b>Lecture Notes</b> (Indicated as “Text 1,2, etc.”) As indicated for each lecture, below
Recommended Reading	<a href="#">Kumar and Varaya</a> Stochastic Systems (1986)  Kalman Filtering <a href="#">Various Files</a>  Stengel RF Optimal Control and Estimation Dover Publications, Inc., NY, 1994

### Course Objectives

This course aims to provide students with a comprehensive understanding of the principles and techniques involved in the analysis, modeling, filtering, and prediction of dynamic systems, enabling them to make informed decisions in the field of system identification..

### Course Content

Dynamic Systems, Convolution, ZOH, Unit Pulse Response, Random Numbers, Variables, and Processes. Correlation and Covariance, Continuous-Discrete Time LTI Systems, Frequency Domain and Relationship Between Fourier Transforms, Approximation of Autocorrelation and Periodogram, The Prediction Process and Transfer-Function Identification, Q-Domain Properties, Four Major Models, and One-step Prediction, Principal Component Analysis, State-Space Representation: Noiseless Data, Linear Least-Squares Regression and the Bayes Theorem, Kalman Filter

### Course Learning Outcomes

1. Students understand the basic principles of system identification and learn its importance.
2. Students learn the basic tools, techniques and principles of system identification.
3. Students learn the applications of system estimation.
4. Students process digital signals with computer-based tools.
5. Students learn stochastic analysis.

### Program Outcomes (PO) Relevant to the Course

4. PO-2.1) Ability to define, formulate and solve complex engineering problems.
5. PO-2.2) Ability to choose and apply appropriate analysis and modeling methods for this purpose.
6. PO-3.1) Ability to design a complex system, process, device or product to meet specific requirements under realistic constraints and conditions.
7. PO-3.2) An ability to apply modern design methods for this purpose.
8. PO-4.1) Ability to develop, select and use modern techniques and tools necessary for the analysis and solution of complex problems encountered in engineering applications.
12. PO-5.3) Ability to collect data to examine complex engineering problems or discipline-specific research topics

### Course Learning Outcomes & Program Outcomes Matrix

<u>CLO</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>
PO-4	3	5	3	4	5
PO-5	3	3	5	5	5
PO-6	3	3	5	5	5
PO-7	3	3	3	5	5
PO-8	3	3	5	5	5
PO-12	3	3	5	5	5

Welcome to Class everyone.

Before starting this semester, it may be wise to be advised about the course organization so as to maximize your classroom performance and get the most fun out of the experience.

To begin, please follow the steps below.

1. Go to <http://avesis.yildiz.edu.tr/kamuran/>
2. Click on My Shares (**Duyuru ve Dokümanlar**)
3. Find the file “**Rules and Guidelines**”
  - a. Download & save and print it
  - b. Read, understand, and even memorize its contents.
  - c. Pay particular attention to **Homework Policy** and **Grading Policy**
4. On the same page, find the file “**YourCourseCode\_Syllabus\_Semester\_Year\_#**”.
  - a. The file will be updated on a weekly basis.
  - b. Make sure you read the assignment(s) **BEFORE** coming to each class, **STARTING THIS WEEK!**
  - c. Download and save any **Textbook(s)** that is (are) posted.
5. It is highly recommended that you
  - a. Read the assigned chapter
  - b. Solve the example and homework problems
  - c. Prepare your questions for me

**BEFORE COMING TO CLASS.**

6. There will be a quiz assignment every week.

I am looking forward to an educational and productive semester.

Good luck to all.

KAK

## HOMEWORK and GRADING POLICY

1. There will be a quiz every week, except the first week, the weeks before the exams, and the holiday weeks (if any) for a maximum total of 11 HWs.
2. Overall quiz grade counts for 35% of final grade.
3. You can follow up weekly grades from [BME33310 GRADES](#), check out the solution sheet from the updated syllabus, and see the highest and lowest scoring submissions when the link on the relevant week becomes active.
4. Midterm and Final exam grades count for 25% and 40% of the final grade, respectively.

Overall Homework Grade	%35
Midterm	%25
Final	%40

WK	HR	SUBJECT
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I

- I. **WELCOME TO CLASS**
  1. Homework, Grading Policies, General Principles
  2. Class Rules [Classroom Rules](#)
- II. **Identification Types**
  1. Prediction (of Output)
  2. Deconvolution (Diagnosis of Input)
  3. System Identification (of Transfer Function)
- III. **Random Numbers (Scalars)Vectors**

[Text2](#)

[Class Notes Ch 1 4: Statistical Concepts Recap](#)

[Class Notes Ch 1 5: Statistical Examples](#)

$$\text{Variable } \vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} \quad \text{Variable values } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

1. Covariance

$$\mathbf{C}_{\mathbf{w}\mathbf{x}}[k, m] \equiv \mathbb{E}\{(\mathbf{w}[k] - \bar{\mathbf{w}}[k])(\mathbf{x}[k - m] - \bar{\mathbf{x}}[k - m])^T\}$$

$$\mathbf{C}_{\mathbf{w}\mathbf{x}}[k, m] \neq \mathbf{C}_{\mathbf{x}\mathbf{w}}[k, m]$$

2. Covariance Matrix

$$[\Sigma_{\tilde{\mathbf{X}}}]=\mathbb{E}\left[(\vec{X}-\vec{\bar{x}})(\vec{X}-\vec{\bar{x}})^T\right]$$

$$\Sigma_{\tilde{\mathbf{X}}}{}_{ij}=(\Sigma_{\tilde{\mathbf{X}}})_{ij}=\rho_{ij}\sigma_{xi}\sigma_{xj}$$

3. Correlation

$$\mathbf{R}_{xx}=\mathbb{E}\left[\vec{X}^T\vec{X}\right]$$

$$\mathbf{R}_{\mathbf{w}\mathbf{x}}[k, m] \equiv \mathbb{E}\{\mathbf{x}^T[k]\mathbf{w}[k - m]\} \neq \mathbf{R}_{\mathbf{x}\mathbf{w}}[k, m]$$

4. From Covariance to Correlation Coefficient

$$\mathbf{R}_{xy}[k, m] = \frac{\mathbf{C}_{xy}[k, m]}{\sqrt{\sigma_x^2 \sigma_y^2}}$$

5. Expectation Operator
6. Power Spectrum

#### IV. Convolution

##### 1. LTI Systems

[Class Notes Ch 1 3: LTI Systems Recap](#)

- Linear System: Superposable and Scalable
- Time-invariant System

##### 2. Delta-combing a Function/System

##### 3. Continuous-time Convolution

[Class Notes Ch1 2: Discrete Time Recap](#) (Section II)

#### IV. Study Discrete-time Convolution

Follow solution of [Class Notes Ch1 2: Discrete Time Recap](#) (Section XI)

$$y[n] = u[n] + 2y[n - 1] - 3y[n - 2]$$

$$u[n] = [0 \ 1 \ 1 \ -1]$$

#### V. Time Domain Identification

[Class Notes Ch 5: Impulse Response Identification](#) (Section 5)

##### 1. Zero-Order Hold (ZOH)

##### 2. Unit Pulse Response

$$y(kT_s) = \int_{\tau=0}^{\infty} g(\tau) \delta(kT_s - \tau) d\tau$$

- The value of  $g(t)$  for the entire unit pulse is equal to  $g(kT_s)$

$$g_{T_s}[k] = \int_{\tau=(k-1)T_s}^{kT_s} g(\tau) d\tau$$

- The unit pulse remains constant inside  
the interval  $kT_s \leq t < (k+1)T_s$

$$\int_{\tau=kT_s}^{(k+1)T_s} \delta(kT_s - \tau) d\tau = \sum_{n=1}^{\infty} \delta[k - n] = u[k]$$

- Combining the two expressions

$$y(kT_s) = \sum_{n=1}^{\infty} g_{T_s}[n] \delta[k - n]$$

#### V. Systems with Noise

##### Unit-Pulse-Response (UPR) Identification

Read: [Text2](#)

##### a. The importance of UPR

Listening Assignment: [Movie2.2](#) (19:59)

##### b. Direct approach to finding UPR

Listening Assignment: [Movie2.3](#) (25:26)

## V. CLASS OUTLINE for upcoming WEEKS

1. Time Domain Identification
2. Frequency Domain Identification
3. Transfer Function Identification
4. State-Space Identification
5. Kalman Filtering

## VI. HOMEWORK 1

Study well the course prerequisites below for the upcoming quiz:

### 1. BASIC CONCEPTS RECAP

[Chapter 1](#): Concepts

[Chapter 2](#): Mathematical and Physical Concepts

[Chapters 3 and 4](#): First Order Systems- Analogies, Examples

- a. Linear Algebra
  - a. Matrix: Geometric Meaning
  - b. Matrix Operation on Vector
  - c. Eigenvalues and Eigenvectors
- b. Dynamic Systems
  - a. System Behavior/Response/Output
  - b. Transfer Function (Input- -Output)
  - c. Differential Equations and Solutions
  - d. Block Diagrams
    - i. Time Domain
    - ii. Laplace Domain
  - e. Mathematical Representation of Physical Systems

### 2. SYSTEM INPUT TYPES

- a. Delta Dirac
- b. Unit Step
- c. Complex Exponentials (Sinusoidals)

### 3. SIGNALS and SYSTEMS (Watch Dennis Freeman Lectures)

- a. Signal and System Equivalence
- b. Complex Exponentials as LTI System Eigenvectors
- c. Complex Phasors as System Eigenvalues
- d. Bode Plot

4. To learn about [Covariance](#) and [Correlation](#) at a very basic level, follow the hyperlinks (quite silly but illustrative website for 2 short movies of about 20 min long, each)

### 5. System Definitions Recap

[Class Notes Ch1 2: Discrete Time Recap](#) (Section I)

WK	HR	SUBJECT
II		Random Processes

1. Time/Ensemble Average
2. Stationary Processes
  - a. Wide-Sense Stationary

$$\begin{aligned}\mathbb{E}\{\bar{x}\} &= \mathbf{0} \quad \rightarrow \mathbf{R}_{xy}[k, m] = \mathbf{C}_{xy}[k, m] \\ \mathbf{R}_{xy}[k, m] &= \mathbf{R}_{xy}[m] = \mathbb{E}\{\mathbf{x}[k]^T \mathbf{y}[k - m]\}\end{aligned}$$

- b. Strict-Sense Stationary (Ergodic)

3. Autocorrelation Properties for WSS

$$\mathbf{R}_{uu}[\tau = 0] = \mathbb{E}\{\mathbf{u}^T \mathbf{u}\} = \begin{cases} \text{var}(\mathbf{u}) = \sigma_u^2 & \text{for a scalar RV} \\ [\Sigma_Y] = [\Sigma_{\tilde{Y}}] & \text{for a vector RV} \end{cases}$$



## I. RECAP: Stochastic Processes

1. Process, Variable, Number
2. Single Realization vs Ensemble
3. Time- vs. Ensemble Averages
4. Ergodicity

## II. DESIGNING A NOISE FILTER, H

1. Recap from Signals and Systems: [Discrete Time System Description](#)

- a. Difference Equation

$$y[n] = x[n] + p_0 y[n - 1]$$

- b. Polynomial in R

$$\frac{Y}{X} = H(R) = \frac{1}{1 - p_0 R}$$

- d. Infinite Sum (synthetic division of H)

$$H(R) = 1 + p_0 R + p_0^2 R^2 + p_0^3 R^3 + p_0^4 R^4 + \dots + p_0^n R^n$$

### 2. Designing the Filter Coefficients

See [link](#) for details. Study examples well

$$C_v[k, m] = \mathbb{E}\{v[k]v[k - m]\}$$

$$R_v[m] = \sigma^2 \sum_{n=0}^{\infty} h[n]h[n - m]$$

In summary,, by designing the coefficients of the filter  $h[k]$ , we can shape the auto-covariance of  $v[k]$ .

## III. IDENTIFYING A SYSTEM USING CORRELATIONS (See [link](#) for details)

- If  $u[k]$  is selected as White Noise

$$R_u[\tau] = \sigma_u^2 \delta[\tau] = \sigma_u^2 \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

- And if  $R_{yu}[\tau]$  can be computed precisely,

$$R_{yu}[\tau] = \sum_{m=1}^{\infty} g_0[m] R_u[m - \tau]$$

### 1. Identifying a System using Correlations

$$g_0[\tau] = \frac{R_{yu}[\tau]}{\sigma_u^2}$$

### 2. Calculating $R_{uy}[\tau]$ , the cross-correlation of input and output (Both WSS)

$$R_{uy}[\tau] = h[-\tau] * R_{uu}[\tau]$$

### 3. Calculating $R_{yy}[\tau]$ the auto-correlation of output

$$R_{yy}[\tau] = g[\tau] * R_{uy}[\tau]$$

### 4. [Combining it all](#)

$$R_{yy}[\tau] = g[\tau] * g[-\tau] * R_{uu}[\tau]$$

We will use this important relationship when we move on to the Frequency Domain Identification topic.

## IV. APPROXIMATIONS on the AUTOCORRELATION (See link for details)

Also, this formula requires Ensemble Averaging infinite number of simultaneous experiments ( $m = 1 \rightarrow \infty$ ). Therefore, we must use estimates from the average of a limited number of experiments

$$\hat{R}_u^N[\tau] = \frac{1}{N} \sum_{k=0}^{N-1} u[k] u[k - \tau]$$

$$\hat{R}_{yu}^N[\tau] = \frac{1}{N} \sum_{k=0}^{N-1} y[k] u[k - \tau]$$

$$\overrightarrow{g_0[m]} \cong \left[ \hat{R}_u^N[m - \tau] \right]^{-1} \overrightarrow{\hat{R}_{yu}^N[\tau]}$$

The Fourier Transforms of  $R_{xx}[\tau]$  and  $R_{yy}[\tau]$  become their respective Power Spectra. The Fourier Transform of  $\{h[\tau] * h[-\tau]\}$  becomes the DTFT of the system.

## V. EXAMPLES (See [link](#) for details)

1. Noise Filter 1
2. Noise Filter 2 (Matlab)
3. Noise Filter 3 (Ljung)

WK	HR	SUBJECT
IV		

WK	HR	SUBJECT
V		INTRODUCTION to FREQUENCY DOMAIN

## I. OPERATORS and FILTERS

### 1. Forward-shift Operator

$$q^{-1}\{u[k]\} = u[k - 1]$$

### 2. Filters

#### a. System Filter

$$y[k] = \left( \sum_{m=1}^{\infty} g_0[m] q^{-m} \right) u[k]$$

$$G(q) = \sum_{m=1}^{\infty} g_0[m] q^{-m}$$

#### b. Noise Filter

$$v[k] = \left( \sum_{m=1}^{\infty} h[m] q^{-m} \right) e[k]$$

$$H(q) = \sum_{m=1}^{\infty} h[m] q^{-m}$$

#### c. System Output with Filters

$$y[k] = \sum_{m=1}^{\infty} g_0[m] u[k - m] + \sum_{m=0}^{\infty} h[m] e[k - m]$$

$$y[k] = G(q) u[k] + H(q) e[k]$$

## II. FOURIER ANALYSIS

### 1. Fourier Series and Transforms (Click on links and study examples well)

		<b>Periodic</b> in Time Discrete in Frequency	Aperiodic in Time <b>Continuous</b> in Frequency
		Continuous Time Fourier Series ( <a href="#">FS</a> )	Continuous Time Fourier Transform ( <a href="#">FT</a> )
Continuous in Time Aperiodic in Frequency	Analysis Equation	$a_k = \frac{1}{T} \int_0^T x(t) e^{-j\frac{2\pi}{T}kt} dt$ $\omega = \frac{2\pi(rad)}{T(sec)} k$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ $\omega = \omega \left( \frac{rad}{sec} \right)$
	Synthesis Equation	$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
		Discrete Time Fourier Series ( <a href="#">DTFS</a> )	Discrete Time Fourier Transform ( <a href="#">DTFT</a> )
Discrete in Time <b>Periodic</b> in Frequency	Analysis Equation	$a_k = a_{k+N} = \sum_{n=1}^N x[k] e^{-j\left(\frac{2\pi}{N}k\right)n}$ $\omega = \frac{2\pi(rad)}{N(sec)} k$	$X[e^{j(\Omega+2\pi)}] = X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}$ $\omega = \frac{\Omega(rad)}{T_s(sec)}$
	Synthesis Equation	$x[k] = x[k + N] = \frac{1}{N} \sum_{n=0}^{N-1} a_k e^{j\left(\frac{2\pi}{N}k\right)n}$	$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega k} d\Omega$

### 2. Define System Filter as DTFT

(DTFT: Discrete Time Fourier Transform of  $g[m]$ )

(Requires collection of infinite data points)

$$G(e^{j\omega}) = \sum_{m=-\infty}^{\infty} g[m] e^{-j\omega m}$$

WK	HR	SUBJECT
VI		FOURIER ANALYSIS ( <a href="#">Summary</a> )

[Focus](#) on DTFS (DFT and FFT) Twiddle Factors

FILTERS, ETFT and PERIODOGRAMS (For details, click [here](#))

**Sinusoidal Input without Measurement Noise**

$$\begin{aligned}
 u[k] &= \cos \omega k \\
 y[k] &= \sum_{m=1}^{\infty} g[m] u[k-m] \\
 y[k] &= \Re \left\{ e^{j\omega k} \sum_{m=1}^{\infty} g[m] [e^{-j\omega m}] \right\} \\
 y[k] &= |G(e^{j\omega})| \cos(\omega k + \varphi)
 \end{aligned}$$

**Empirical Transfer Function Estimates (ETFE)**

Based on DFT (estimate of DTFT) of Input and Output

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{k=1}^N u[k] e^{-j\omega k}$$

$$U_N[n] = \frac{1}{\sqrt{N}} \sum_{k=1}^N u[k] e^{-j\frac{2\pi k}{N}n}$$

$$\hat{G}(e^{j\omega}) = \frac{Y(e^{j\omega})}{U(e^{j\omega})}$$

$$\hat{G}(e^{j\omega}) = \frac{1}{\sqrt{N}} \sum_{m=1}^N y[m] e^{-j\omega m} / \frac{1}{\sqrt{N}} \sum_{m=1}^N u[m] e^{-j\omega m}$$

$$\hat{G}(e^{j\omega}) = \sum_{m=1}^N y[m] e^{-j\omega m} / \sum_{m=1}^N u[m] e^{-j\omega m}$$

## Properties of ETFE

- ETFE is an asymptotically unbiased estimate
- The variance of ETFE decreases as N increases, but the SNR has a lower-bound
- Estimates of ETFE at neighboring frequencies are not correlated (they are asymptotically uncorrelated)

## II. Periodogram of $\mathbf{u}[k]$

(Squared Magnitude of the DFT of a Signal)

$$|\mathbf{U}_N(\omega)|^2 = \left| \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathbf{u}[k] e^{-j\omega k} \right|^2$$

### a. Definition 2: Quasi-Stationary Autocorrelation Function

$$\mathbf{R}_s\{\tau\} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbf{R}_s\{k, k - \tau\} \quad \forall t$$

$$\mathbb{E}\{\mathbf{u}[k]\mathbf{u}[k - \tau]\} = \mathbf{R}_u\{k, k - \tau\}$$

$$\mathbf{R}_u\{\tau\} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N \mathbb{E}\{\mathbf{u}[m]\mathbf{u}^T[m - \tau]\}$$

### b. Power Spectrum of Quasi-Stationary Signals

$$\mathbf{R}_u[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega = \mathbb{E}\{\mathbf{u}^2[k]\}$$

$$\Phi_{uw}(\omega) = \sum_{\tau=-\infty}^{\infty} \mathbf{R}_{uw}[\tau] e^{-j\omega\tau}$$

### c. Problems with Periodogram-Spectrum Relationship

$$\Phi_y(\omega) = \sum_{\tau=-\infty}^{\infty} \mathbf{R}_y[\tau] e^{-j\omega\tau}$$

For WSS White input:

$$\Phi_y(\omega) = |G(e^{j\omega})|^2 \sigma^2$$

$$|\mathbf{Y}_N(\omega)|^2 = \left| \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathbf{y}[k] e^{-j\omega k} \right|^2$$

#### d. Smoothed Periodograms

Periodogram is an **asymptotically unbiased estimator** of the Spectrum.

Variance (scatter around mean) does not decrease as more data are collected.

In addition, the larger the output, the larger the variance.

Error estimates at all frequencies are correlated.

#### e. Solution: Welch's Method

Collect large amount of data  $\mathbf{y}[k]$ ,  $1 \leq k \leq N$

Break the data set in 2, 4, 8, 16, ... chunks, say into  $n$  segments of  $m$  data points, each, so that  $nm = N$  and

$$\mathbf{y}^p[k] = \mathbf{y}[k + m(p-1)] \text{ for } 1 \leq k \leq m-1 \text{ and } 1 \leq p \leq n$$

Form  $n$  Periodogram-based spectral estimates

$$\Phi_m^p(\omega) = \frac{1}{m} \left| \sum_{k=1}^{m-1} \mathbf{y}^p[k] e^{-j\omega k} \right|^2 \quad 1 \leq p \leq n$$

Find the estimate in each  $\Phi_m^p(\omega)$

$$\text{var}\{\hat{\Phi}_m^A(\omega)\} = \frac{1}{n} \text{var}\{\hat{\Phi}_m^p(\omega)\} = \frac{1}{n} \text{var}\{\hat{\Phi}_N(\omega)\}$$

so that the variance of  $\hat{\Phi}_N(\omega)$  is down-scaled by a factor of  $n$ , since

$$\text{var}\{\hat{\Phi}_m^p(\omega)\} = \text{var}\{\hat{\Phi}_N(\omega)\}$$



Unfortunately, as  $n \uparrow$ ,  $m \downarrow$  (since  $N$  is fixed), the number of frequency points investigated gets smaller as  $m$  gets smaller.  
 The result is better estimates at fewer frequencies, which may be a problem if the Spectrum-Frequency plot has “narrow” peaks.  
 But the method works well for long data streams (called “excess data”)

### III. FREQUENCY FILTERING

The choice of window (main lobe,  $\gamma$ ) width is much more important than which window to choose.

Need a good idea of what spectrum looks like to select  $\gamma$   
**because there will be a trade-off:**

Small  $\gamma \Rightarrow \begin{cases} \text{Large lobe} \Rightarrow \downarrow \text{variance} \Rightarrow \text{smooth } \hat{\Phi}_y \\ \text{averaging more data within window} \Rightarrow \text{peaks lost} \Rightarrow \text{bias} \end{cases}$

#### Rectangular Weighing Function

$$\overline{\hat{\Phi}_y}(\omega_k) = \frac{1}{2m+1} \sum_{j=-m}^m \hat{\Phi}_y(\omega_{k-j})$$

$$\omega_\gamma(\omega) = \frac{1}{2m+1}$$

#### Blackman-Tukey Estimate

Given some data,  $y[k]$ , that can be measured,  
 an auto-correlation estimate can be formed

$$\hat{R}_y^N(\tau) = \frac{1}{N} \sum_{k=\tau}^{N-1} y[k]y[k-\tau]$$

after getting the estimate,  
 its DTFT is formed

$$\hat{\Phi}_y(\omega) = \sum_{\tau=-\infty}^{\infty} \hat{R}_y[\tau] e^{-j\omega\tau}$$

$$\hat{\Phi}_{BT}(\omega) = \sum_{\tau=-(\gamma-1)}^{\gamma-1} \omega_\gamma[\tau] \hat{R}_y[\tau] e^{-j\omega\tau}$$

- V. More [Details](#) on Periodogram, Power Spectral Density, Smoothing, Frequency-windowing
- VI. HOMEWORK: STUDY EXAMPLES in BOOK and RECREATE the CODES for ETFE and PERIODOGRAMS

1. In [Plett's](#) Book
  - a. Study Section 3.6\_Smoothed Periodograms. Understand the examples well.
  - b. Study Section 3.7\_Frequency Filtering. Understand the examples well.
  - c. Study Section 3.8\_Blackman-Tuckey Estimate. Understand the various windows well.
2. In Section 3.9\_ETFE and SPA Toolbox Commands: Study the [example](#) problem and from the given state-space representation,
  - a. Reverse-engineer the equations and obtain the schematic of the system with 3 masses, 2 springs and 2 dashpots.
  - b. Create the model in Simulink.
  - c. Write the differential equations in state-space format
  - d. Understand the solution of the problem
  - e. Explain the relevant commands in the code
  - f. Run the code in Matlab and reproduce the plots

VII [Details](#) of Lecture Sections

I. Transfer Function Identification ([Text4](#))

1. Discrete-time Transfer Function Models

$$y[k] = G(q) u[k] + H(q) e[k]$$

$$G(q) = \sum_{k=1}^{\infty} g[k] q^{-k}$$

$$H(q) = \sum_{k=0}^{\infty} h[k] q^{-k}$$

2. Generalized Polynomial Representation of System TF

$$G(q) = q^{-n_k} \frac{b_1 q^{-1} + \dots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a}}$$

Total of  $n_b + n_a + 3$  unknowns to define the TF.

In mathematical notation:  $\dim\{n_a, n_b, n_k\} = 3$

II. Correspondence: Discrete/Continuous-time signals

1. Damped Sinusoid: Discrete-time

2. Damped Sinusoid: Continuous-time

III. Analogy between continuous and discrete time-frames

1. Discretization of a Continuous Signal (N)

2. Effect of Sampling Rate on the Discrete Signal

- a. Magnitude
- b. Phase Angle (Argument)
- c. Pole Location

IV. Lines of Constant N in the q-Plot

V. Some q-Domain Pole Locations

VI. q- vs. s-Domain corresponding pole locations

VII. q-Plane Constant Loci Equivalence of Time-Domain Performance Parameters

$$\zeta, \omega_n, \omega_d, \text{ and } \sigma. = f(r, \theta)$$

$$\omega_n = \frac{\sqrt{\theta^2 + \ln^2(r)}}{T_s}, \quad \omega_d = \frac{1}{T_s} \theta, \quad \sigma = \frac{1}{T_s} \ln[r], \quad \zeta = -\frac{\ln[r]}{\sqrt{\theta^2 + \ln^2(r)}}$$

## VIII. Plotting Constant $\zeta$ Lines in q-Plane

If we increase  $\sigma$  along a constant  $\zeta$  line in the q-plane, the radius shrinks while the angle sweeps counterclockwise

## IX. Plotting Constant $\omega_n$ Lines in q-Plane

Constant  $\omega_n$  lines asymptotically approach the constant  $\omega_d$  lines as we get closer to the unit circle (i.e, as  $r \rightarrow 1$ , which means as  $\zeta \rightarrow 0$ ).

## X. Expressing q in terms of $\omega_n$ and $\zeta$

- a. Undamped case
- b. Critically damped case
- c. Underdamped case ( $0 < \zeta < 1$ )

## XI. Nyquist-Shannon Sampling Theorem (from Wikipedia):

## XII. [EXAMPLES and EXERCISES](#)

- 1. Exponential (Geometric Series)
- 2. Case 3: Damped Sinusoid
- 4. Inverse q-domain Examples
- 5. [HW 6 Solution](#)

## VIII. TRANSFER SYSTEM IDENTIFICATION ([Details](#))

### I. Four Major Models (OE, ARX, ARMAX, BJ) [Movie4.4](#) (21:27)

- 1. Output Error
- 2. ARMAX
- 3. Box Jenkins
- 4. ARX

### II. The Prediction Process

- 1. Auto Regressive Moving Average (ARMA,  $n_a \neq 0$  and  $n_c \neq 0$ )
- 2. Moving Average (MA,  $n_a = 0$ )
- 3. Auto Regressive (AR,  $n_c = 0$ )

### III. One-step Prediction

#### a. Monic Denominator A (a<sub>0</sub>=1): ARX Model

$$\hat{y}_p[k|k-1] = \{1 - \mathbf{A}(q, \theta)\} y[k] + \mathbf{B}(q, \theta) u[k]$$

$$\begin{aligned} \hat{y}_p[k|k-1] = & -a_1 y[k-1] - \dots - a_{n_a} y[k-n_a] \\ & + b_0 u[k] + b_1 u[k-1] + \dots + b_{n_b} u[k-n_b] \end{aligned}$$

#### b. Input Vector (Feed-forward, $y[k-n_a]$ , and Feed-back, $u[k-n_b]$ , Inputs)

$$x[k] = \begin{Bmatrix} -y[k-1] - \dots - y[k-n_a] \\ +u[k] + u[k-1] + \dots + u[k-n_b] \end{Bmatrix}^T$$

#### c. Filter Coefficients ( $n_a$ and $n_b$ indicate order of filters)

$$\theta^T = [a_1 \dots a_{n_a} \ b_0, b_1 \dots b_{n_b}]$$

### IV. Filter Characteristics

#### a. Input-Output Relationship (A=1)

$$y[k] = \mathbf{B}(q, \theta) u[k] + e[k]$$

#### b. Time-Series/Single Process (B=0, No input)

$$y[k] = \{1 - \mathbf{A}(q, \theta)\} y[k] + e[k] \quad \text{True}$$

$$\hat{y}[k|k-1] = -a_1 y[k-1] - \dots - a_{n_a} y[k-n_a] \quad \text{Predicted}$$

$$\begin{aligned} \hat{y}[k|k-1] &= x^T[k] \theta[n_a] = \theta^T[n_a] x[k] \\ &= \sum_{l=1}^{n_a} a^T[l] y[k-l] = \sum_{l=1}^{n_a} a_l y_{k-l} \end{aligned}$$

$$= [a_1, a_2, a_3, \dots, a_{n_a}] \begin{bmatrix} -y_{k-1} \\ -y_{k-2} \\ -y_{k-3} \\ \vdots \\ -y_{k-n_a} \end{bmatrix}$$

**c. Time Series/Multiple Processes (Ensemble, n=1 ....N)**

$$\begin{aligned} \hat{y}_1[k|k-1] &= -a_1 y_1[k-1] - \dots - a_{n_a} y_1[k-n_a] \\ \hat{y}_2[k|k-1] &= -a_1 y_2[k-1] - \dots - a_{n_a} y_2[k-n_a] \\ &\vdots \\ \hat{y}_N[k|k-1] &= -a_1 y_N[k-1] - \dots - a_{n_a} y_N[k-n_a] \end{aligned}$$

$$\begin{bmatrix} -y_{1,k-1} & -y_{1,k-2} & \dots & -y_{1,k-n_a} \\ -y_{2,k-1} & -y_{2,k-2} & \dots & -y_{2,k-n_a} \\ -y_{3,k-1} & -y_{3,k-2} & \dots & -y_{3,k-n_a} \\ \vdots & \vdots & \vdots & \vdots \\ -y_{N,k-1} & -y_{N,k-2} & \dots & -y_{N,k-n_a} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n_a} \end{bmatrix} = \begin{bmatrix} \hat{y}_1[k|k-1] \\ \hat{y}_2[k|k-1] \\ \hat{y}_3[k|k-1] \\ \vdots \\ \hat{y}_N[k|k-1] \end{bmatrix}$$

$$\mathbf{x}[N \times n_a] \cdot \boldsymbol{\theta}[n_a \times 1] = \hat{\mathbf{y}}[N \times 1]$$

$$a_1 \begin{bmatrix} -y_{1,k-1} \\ -y_{2,k-1} \\ -y_{3,k-1} \\ \vdots \\ -y_{N,k-1} \end{bmatrix} + a_2 \begin{bmatrix} -y_{1,k-2} \\ -y_{2,k-2} \\ -y_{3,k-2} \\ \vdots \\ -y_{N,k-2} \end{bmatrix} + \dots + a_{n_a} \begin{bmatrix} -y_{1,k-n_a} \\ -y_{2,k-n_a} \\ -y_{3,k-n_a} \\ \vdots \\ -y_{N,k-n_a} \end{bmatrix} = \begin{bmatrix} \hat{y}_1[k|k-1] \\ \hat{y}_2[k|k-1] \\ \hat{y}_3[k|k-1] \\ \vdots \\ \hat{y}_N[k|k-1] \end{bmatrix}$$

**d. Non – square Input Matrix ( $N \neq n_a$ )**

If  $\boldsymbol{\theta}[n_a \times 1]$  does not span the Column Space of  $\mathbf{x}[N \times n_a]$   
Then calculate optimal  $\boldsymbol{\theta}[n_a \times 1]$   
that gives best prediction of output  $\hat{\mathbf{y}}[N \times 1]$

**V. Least Squares Estimation**

- a. Initial model structure selection  
[Movie4.5](#) (18:20)
- b. Fitting parameterized model  
[Movie4.6](#) (20:14)

- c. Fitting parameterized model: Cost function  
[Movie4.7](#) (13:51)
- d. Solving the linear least-squares ARX problem  
[Movie4.8](#) (19:18)

## VI. Cost Function

- e. Fitting parameterized model: Cost function  
[Movie4.10](#) (18:15)

$$\mathbf{y}_N = \sum_{k=1}^N \mathbf{y}[k]^2 \quad \text{Output (Scalar)}$$

$$\mathbf{R}_{xx}^N = \sum_{k=1}^N x[k]x[k] = \quad x x^T: \text{Autocorrelation of Input}$$

$$\mathbf{R}_{xy}^N = \sum_{k=1}^N \varphi[k] \mathbf{y}[k] = \quad x^T \mathbf{y}: \text{I/O Crosscorrelation}$$

Cost Function

$$\begin{aligned} V_N(\theta) &= \sum_{k=1}^N \epsilon^2[k; \theta] \\ &= \sum_{k=1}^N \{\mathbf{y}[k] - \theta^T x[k]\}^2 \\ &= \mathbf{y}_N - 2\theta^T \mathbf{R}_{xy}^N + \theta^T \mathbf{R}_{xx}^N \theta \end{aligned}$$

## VII. Optimal Solution to the LSE Problem

### a. Linear Algebra

$$V_N(\theta) = \mathbf{y}_N - \mathbf{f}_N^T \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy} + (\theta - \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy})^T \mathbf{R}_N (\theta - \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy})$$

$$\mathbf{y}_N - \mathbf{f}_N^T \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy} > 0$$

$$\min (\theta - \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy})^T \mathbf{R}_N (\theta - \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy}) \Rightarrow \theta = \mathbf{R}_{xx}^{-1} \mathbf{R}_{xy}$$

See [Class Notes SID 7-1: TFID Problems and Examples](#)

### b. Calculus (Derivative):

To find  $\hat{\theta}$  ( $\theta_{opt}$ ), which minimizes  $V_N(\theta)$ , take a derivative

$$\frac{d}{d\theta} V_N(\theta) = -2\mathbf{R}_{xy}^N + 2\mathbf{R}_{xx}^N \hat{\theta} = 0$$

$$\mathbf{R}_N \hat{\theta} = \mathbf{R}_{xy}^N \quad \text{or} \quad \mathbf{x} \mathbf{x}^T \hat{\theta} = \mathbf{x}^T \mathbf{y}$$

$$\hat{\theta} = \mathbf{R}_{xx}^N \mathbf{R}_{xy}^N \quad \text{or} \quad \hat{\theta} = (\mathbf{x} \mathbf{x}^T)^{-1} \mathbf{x}^T \mathbf{y}$$

Remembering that, by definition

$$\hat{\mathbf{y}} = \mathbf{x}^T \hat{\theta} = \mathbf{x}^T [(\mathbf{x} \mathbf{x}^T)^{-1} \mathbf{x}^T \mathbf{y}]$$

$$[\mathbf{x}^T (\mathbf{x} \mathbf{x}^T)^{-1} \mathbf{x}^T] \mathbf{y} = \hat{\mathbf{y}}$$

### c. Some terminology (definitions for $\mathbf{x} \mathbf{x}^T \hat{\theta} = \mathbf{x}^T \mathbf{y}$ )

$$\mathbf{x} \mathbf{x}^T \quad \triangleq \mathbf{N} \text{ Normal Matrix}$$

$$(\mathbf{x} \mathbf{x}^T)^{-1} \quad \triangleq \text{Cofactor Matrix of } \theta \text{ (similar to covariance)}$$

$$\triangleq \text{Pseudo - inverse (Moore - Penrose) of } \mathbf{N}$$

$$\mathbf{x}^T \mathbf{y} \quad \triangleq \text{Moment Matrix (of regressors and regressand)}$$

$$[\mathbf{x}^T (\mathbf{x} \mathbf{x}^T)^{-1} \mathbf{x}^T] \triangleq \text{Hat Matrix (Puts a hat on } \mathbf{y})$$

Also called "Projection Matrix of  $\varphi$  onto the column space of  $\varphi$ "



d. Geometry (Orthogonality Principle)

$$\epsilon_{opt}[k] = y[k] - \theta_{opt}^T x[k]$$

$$\epsilon_{opt}[k] \triangleq \hat{\epsilon}[k] \perp \hat{y}[k] = y_{opt}[k]$$

(Estimation error is uncorrelated with the estimate!)

And since  $\hat{y}[k]$  is co-planar with  $\varphi[k]$

$$x[k] \perp \epsilon_{opt}[k] \Rightarrow \mathbb{E}\{x\epsilon_{opt}\} = 0$$

(Estimation error is uncorrelated with past samples!)  
These are **INNOVATIONS** (See next week, II-1)

$$\mathbb{E}\{x\epsilon_{opt}\} = \mathbb{E}\{x[y - x^T \hat{\theta}]\} = \mathbb{E}\{xy - xx^T \hat{\theta}\} = 0$$

$$= \mathbb{E}\{xy\} - \mathbb{E}\{xx^T\} \hat{\theta} = 0$$

$$\mathbf{R}_{xx}^N \hat{\theta} = \mathbf{R}_{xy}^N$$

VIII. Examples (See [Class Notes 7.0](#) and [7.1](#))

- a. Linear Input-Output (Constant velocity)
- b. Non-linear Input-Output (Free fall)

WK	HR	SUBJECT
VIII		Holiday

WK	HR	SUBJECT
IX		MIDTERM EXAM

Open notes  
Laptop with Matlab, no internet

Reading/Listening Assignments for Midterm

1. Introduction to transfer functions  
[Movie4.1](#) (22:08)
2. Time response versus pole locations  
[Movie4.2](#) (35:21)
3. q- and s-plane representation of constant-parameter lines

I. Least Square Estimation  
Class 10 Details

II. Principle Component Analysis

Read [Class Notes](#)

Extra Reading: [1](#), [2](#), [3](#), [4](#)

See last week's syllabus for  $V(\theta)$  expression below, Section VII-a

$$V(\theta) = y - R_{xy}^T R_{xx}^{-1} R_{xy}^T + (\theta - R_{xx}^{-1} R_{xy}^T)^T R_{xx}^N (\theta - R_{xx}^{-1} R_{xy}^T)$$

$$V_{N,min}(\theta) = y - R_{xy}^T R_{xx}^{-1} R_{xy}^T \quad \text{for} \quad \hat{\theta} - R_{xx}^{-1} R_{xy}^T = 0$$

$$V_N(\theta) = V_{N,min}(\theta) + (\theta - \hat{\theta})^T R_{xx}^N (\theta - \hat{\theta})$$

$\theta - \hat{\theta} \triangleq$  Deviation Vector

$$Q_R \triangleq \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1N} \\ q_{21} & q_{22} & \dots & q_{2N} \\ q_{31} & q_{32} & \dots & q_{3N} \\ \vdots & \vdots & \vdots & \vdots \\ q_{N1} & q_{N2} & \dots & q_{NN} \end{bmatrix} \equiv \text{Eigenvector Matrix of } R_{xx}^N$$

$$\Lambda_R \triangleq \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix} \equiv \text{Eigenmatrix of } R_{xx}^N$$

$$R_{xx}^N = Q_R \Lambda_R Q_R^T \text{ (Eigen-decomposition of } R_{xx}^N)$$

$$V_N(\theta) = V_{N,min}(\theta) + (\theta - \hat{\theta})^T (Q_R \Lambda_R Q_R^T) (\theta - \hat{\theta})$$

$$V_N(\theta) = V_{N,min}(\theta) + [(\theta - \hat{\theta})^T Q_R] (\Lambda_R) [Q_R^T (\theta - \hat{\theta})]$$

$$E^T \triangleq (\theta - \hat{\theta})^T Q_R$$

$$V_N(\theta) = V_{N,min}(\theta) + E^T \Lambda_R E$$

$$V_N(\theta) = V_{N,min}(\theta) + \sum_{k=1}^N \lambda_k v_k^2$$

Constant Error Contours =  $f(\lambda_1 v_1^2 + \dots + \lambda_N v_N^2)$  Ellipses

### III. SYSTEM IDENTIFICATION: Transfer Function Methods with Matlab

#### 1. Dryer Example

(Gregory Platt, Chapter 1, Section 2)

- a. [Text](#)
- b. [Movie 4.5](#)
- c. [Class Notes](#): Details for Code

#### 2. [Toolbox Methods](#) for ARX, ARMAX, OE,BJ

- a. [4.11](#): Toolbox methods (1): Frequency response
- b. [4.12](#): Toolbox methods (2): Unit-pulse response, residuals
- c. [4.13](#): Toolbox methods (3): Model validation using correlations
- d. [4.14](#): Toolbox methods (4): ARX model size
- e. [4.15](#): Example with nonwhite noise

### IV. HOMEWORK ([Link](#))

- 1. Readings
- 2. Dryer Example

WK	HR	SUBJECT
XI		KALMAN FILTER
	I.	Linear Least-Squares Regression Re-Cap
		<ol style="list-style-type: none"> <li>1. ARX Model</li> <li>2. Least Squares Estimation</li> <li>3. Diagonalization</li> <li>4. Singular Value Decomposition</li> <li>5. PCA</li> </ol>
	II.	<a href="#">Innovations</a>
	III.	Kalman Filter ( <a href="#">Class Notes</a> )
		<ol style="list-style-type: none"> <li>1. <a href="#">Dan Simon</a></li> <li>2. <a href="#">Bzarg</a></li> <li>3. <a href="#">Various Files</a></li> <li>4. <a href="#">Wikipedia</a></li> </ol>
	IV.	HOMEWORK ( <a href="#">Link</a> )
	1.	Dan Simon
		<p>Read, understand example</p> <p>Write and run your own Matlab code</p> <p>Explain/discuss</p>

WK	HR	SUBJECT
XII		KALMAN FILTER-II

- I. Kalman Recap
- II. Kalman Filter Matlab Code

```
function s = kalmanf(s,u)
    s.x = s.A*s.x + s.B*u;
    s.P = s.A * s.P * s.A' + s.Q;
    % Compute Kalman gain factor:
    K = s.P*s.H'*inv(s.H*s.P*s.H'+s.R);
    % Correction based on observation:
    s.x = s.x + K*(s.z - s.H *s.x);
    s.P = s.P - K*s.H*s.P;
end
```

$$\Sigma_{t|t-1} = F_t \Sigma_{t-1|t-1} F_t^T + Q_t$$

$$K_t = \Sigma_{t|t-1} H_t^T (H_t \Sigma_{t|t-1} H_t^T + R_t)^{-1}$$

$$\begin{aligned} \hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} + K_t (z_t - H_t \hat{\mathbf{x}}_{t|t-1}) \\ \Sigma_{t|t} &= (I - K_t H_t) \left( F_t \Sigma_{t-1|t-1} F_t^T + Q_t \right) \end{aligned}$$

- III. Visit to YTU Physiological Control Lab
- IV. Week 11 [Homework Solution](#)
- V. [Final Exam](#) (Finalized)

WK	HR	SUBJECT
XIII		Class Canceled Instructor out of Town Gather data for Final Exam

WK	HR	SUBJECT
XIV		New Year's Day No class