Outline

- Root finding
  - Bisection method
  - Regula Falsi
  - Secant
  - Fixed point
  - Newton-Raphson
- Solution of linear systems
- Interpolation
- Numerical Integration
- Numerical differantiation
- Finite difference for solving differential equations

- Mid-term exam: %60
- Final: %40

References:

- Steven C. Chapra and Raymond P. Canale, Numerical Methods for Engineers, Mc Graw Hill Education
- Chapra&Canale, çev. Heperkan ve Keskin : Mühendisler için Sayısal Yöntemler, Literatür Yyn.
- Mathews, John H. Numerical Methods for Mathematics, Science and Engineering-Prentice Hall Publ.
- Jeffrey R. Chasnov, Numerical methods, The Hong Kong University of Science and Technology
- Behiç Çağal : Sayısal Analiz, Birsen Yayınevi.
- Bakioğlu, M.; Sayısal Analiz, Beta Yayınları.
- Sayısal analiz ders notları, Zafer Kütüğ

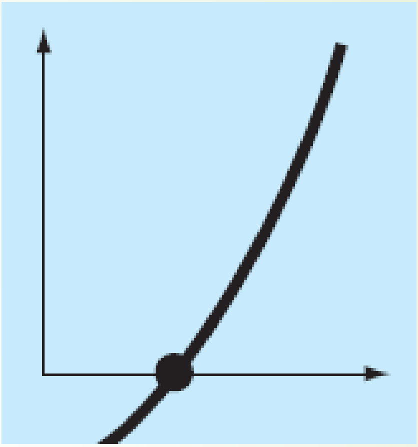
1) Root finding

- What we learned was:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- For solving:

$$f(x) = ax^2 + bx + c = 0$$



- However, there are many other functions for which the root cannot be determined so easily.

1.1) Bisection (Bifurcation method)

- The bisection method is a numerical algorithm used to find the roots of a given function.
- The bisection method, which is alternatively called binary chopping, interval halving, or Bolzano's method, is one type of incremental search method in which the interval is always divided in half.
- **If a function changes sign over an interval, the function value at the midpoint is evaluated**. The location of the root is then determined as lying at the midpoint of the subinterval within which the sign change occurs. The process is repeated to obtain refined estimates.
- The basic idea of the method is to repeatedly divide an interval in half and then select the subinterval in which the root must lie, until the interval becomes sufficiently small to obtain the desired accuracy.

1.1) Bisection (Bifurcation method)

- f(x) function is continious function between [a,b].
- If f(a).f(b)<0, then there must be at least one root that make f(x)=0 between [a,b]
- The error after n iteration is lb-al/2

1.1) Bisection (Bifurcation method)

1. Choose an interval [a,b] such that f(a) and f(b) have opposite signs.
2. Compute the midpoint of the interval: $c = (a + b) / 2$.
3. Evaluate the function at the midpoint: f(c).
4. If f(c) is zero, then c is the root and we are done.
5. Otherwise, if f(c) and f(a) have opposite signs, then the root must lie in the interval [a,c]. We repeat the process on this subinterval.
6. Otherwise, if f(c) and f(b) have opposite signs, then the root must lie in the interval [c,b]. We repeat the process on this subinterval.
7. Repeat steps 2-6 until the interval [a,b] becomes sufficiently small (that satisfies our error limit).

1.1) Bisection (Bifurcation method)

Step 1: Choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)f(x_u) < 0$.

Step 2: An estimate of the root x_r is determined by

$$x_r = \frac{x_l + x_u}{2}$$

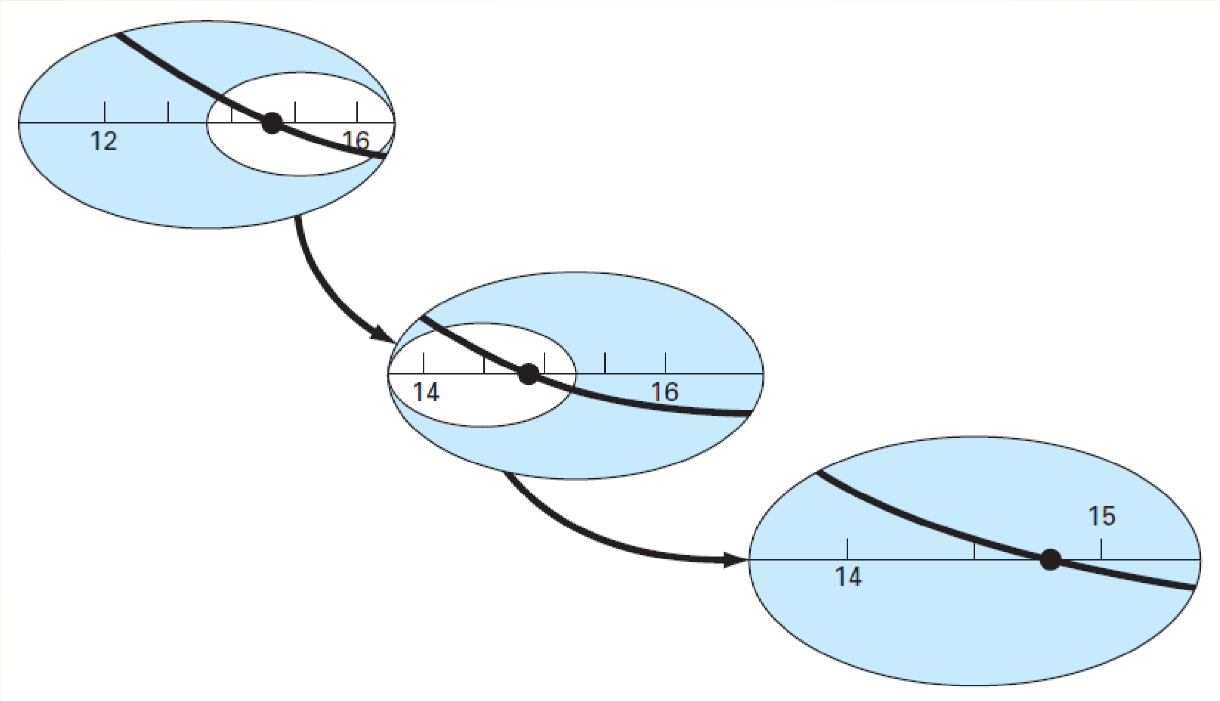
Step 3: Make the following evaluations to determine in which subinterval the root lies:

(*a*) If $f(x_l)f(x_r) < 0$, the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and return to step 2.

(*b*) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval. Therefore, set $x_l = x_r$ and return to step 2.

(*c*) If $f(x_l)f(x_r) = 0$, the root equals x_r; terminate the computation.

1.1) Bisection (Bifurcation method)



1.1) Bisection (Bifurcation method)

- Example: Find the root of $f(x) = x^3 + x^2 - 3x - 3$ between [1,2], $\Delta_x = 1 \cdot 10^{-4}$

n	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\|b-a\|/2^n$	Gerçek hata	Δ_x
1	1	2	1,5	-4,000000	3,000000	-1,875000	0,500000	-0,232051	
2	1,5	2	1,75	-1,875000	3,000000	0,171875	0,250000	0,017949	0,25
3	1,5	1,75	1,625	-1,875000	0,171875	-0,943359	0,125000	-0,107051	0,125
4	1,625	1,75	1,6875	-0,943359	0,171875	-0,409424	0,062500	-0,044551	0,0625
5	1,6875	1,75	1,71875	-0,409424	0,171875	-0,124786	0,031250	-0,013301	0,03125
6	1,71875	1,75	1,734375	-0,124786	0,171875	0,022030	0,015625	0,002324	0,015625
7	1,71875	1,734375	1,726563	-0,124786	0,022030	-0,051755	0,007813	-0,005488	0,0078125
8	1,726563	1,734375	1,730469	-0,051755	0,022030	-0,014957	0,003906	-0,001582	0,0039063
9	1,730469	1,734375	1,732422	-0,014957	0,022030	0,003513	0,001953	0,000371	0,0019531
10	1,730469	1,732422	1,731445	-0,014957	0,003513	-0,005728	0,000977	-0,000605	0,0009766
11	1,731445	1,732422	1,731934	-0,005728	0,003513	-0,001109	0,000488	-0,000117	0,0004883
12	1,731934	1,732422	1,732178	-0,001109	0,003513	0,001201	0,000244	0,000127	0,0002441
13	1,731934	1,732178	1,732056	-0,001109	0,001201	0,000046	0,000122	0,000005	0,0001221
14	1,731934	1,732056	1,731995	-0,001109	0,000046	-0,000532	0,000061	-0,000056	6,104E-05

1.1) Bisection (Bifurcation method)

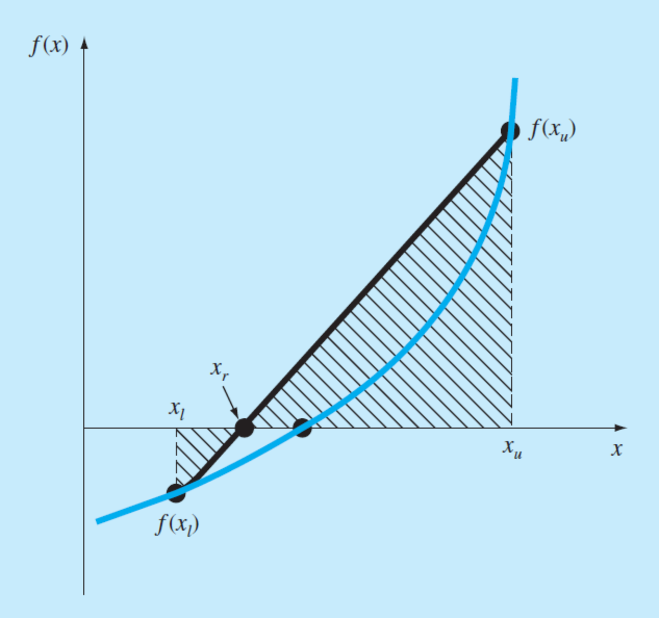
- Example: Your turn?

- $F(x) = 2x^3+\sin^2(x)-5x$

1.2) Regula Falsi Method (False position)

- Although bisection is a perfectly valid technique for determining roots, its "brute-force" approach is relatively inefficient.
- False position is an alternative based on a graphical insight.
- shortcoming of the bisection method is that, in dividing the interval from x_l to x_u into equal halves, no account is taken of the magnitudes of $f(x_l)$ and $f(x_u)$. For example, if $f(x_l)$ is much closer to zero than $f(x_u)$, it is likely that the root is closer to x_l than to x_u
- An alternative method that exploits this graphical insight is to join $f(x_l)$ and $f(x_u)$ by a straight line
- The intersection of this line with the x axis represents an improved estimate of the root.
- The fact that the replacement of the curve by a straight line gives a "false position" of the root is the origin of the name, method of false position, or in Latin, regula falsi.

1.2) Regula Falsi Method (False position)



$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$f(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

Collect terms and rearrange:

$$x_r\,[f(x_l) - f(x_u)] = x_u f(x_l) - x_l f(x_u)$$

Divide by $f(x_l) - f(x_u)$:

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

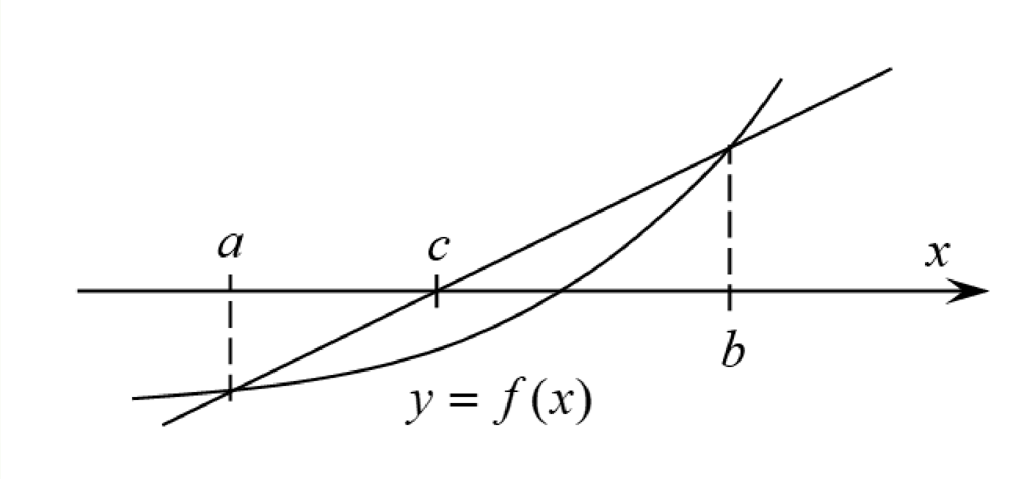
$$x_r = \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$$\boxed{x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}}$$

1.2) Regula Falsi Method (False position)



$$y = f(x) = f(a) + \frac{f(a) - f(b)}{a - b}(x - a)\,; \quad y = 0 \quad \rightarrow \quad \mathrm{x} = c = a - \frac{a - b}{f(a) - f(b)} f(a)$$

- If f(c).f(a)<0, change b with c; use a and c
- If f(c).f(b)<0 change a with c; use b and c

1.2) Regula Falsi Method (False position)

- Example: Using regula falsi method find the root of $f(x) = 3x + \sin(x) - e^x$ between [0,1], $\Delta_x = 1 \cdot 10^{-3}$

n	x_0	x_1	$f(x_0)$	$f(x_1)$	x_2	$f(x_2)$	$\measuredangle x$
1	1	0	1,1231892	-1	0,4709896	0,265158816	
2	0	0,47099	-1	0,265159	0,3722771	0,029533669	-0,09871
3	0	0,372277	-1	0,029534	0,3615977	0,002941	-0,01068
4	0	0,361598	-1	0,002941	0,3605374	0,000289449	-0,00106
5	0	0,360537	-1	0,000289	0,3604331	2,84541E-05	-0,0001

$$\mathrm{x} = c = a - \frac{a-b}{f(a)-f(b)} f(a)$$

1.2) Regula Falsi Method (False position)

- Example: Your turn?

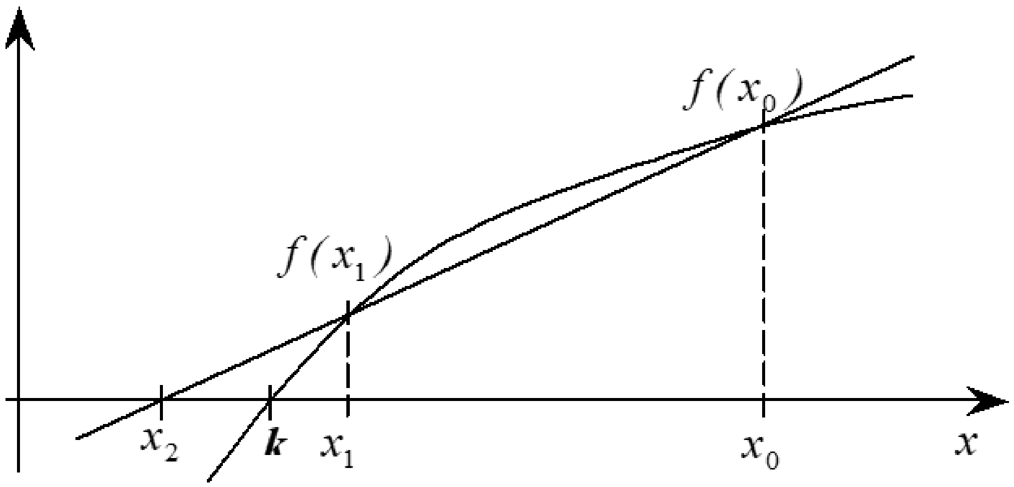
- $F(x) = e^x + x^{(1/3)}$

1.3) Secant method

- The secant method is a numerical root-finding algorithm that is used to find the root of a function. It is a derivative-free method and is based on the idea of using a straight line to approximate the behavior of the function near the root.

- The secant method requires two initial guesses, x_0 and x_1, which should be chosen.

1.3) Secant method (Open method – Not bracketing)



$$\frac{x_1 - x_2}{f(x_1)} = \frac{x_0 - x_1}{f(x_0) - f(x_1)} \quad \rightarrow \quad x_2 = x_1 - f(x_1)\frac{x_0 - x_1}{f(x_0) - f(x_1)}$$

- X_1 is more closer to the root than x_0; ensure $|f(x_1)| < |f(x_0)|$
- This helps make iterations more effectively

1.3) Secant method

- Example 1: Using Secant method, [1,2]

$f(x) = x^3 + x^2 - 3x - 3$ $\Delta_x = 1 \cdot 10^{-4}$

n	x_0	x_1	$f(x_0)$	$f(x_1)$	x_2	$f(x_2)$	x
1	1	2	-4	3	1,5714286	-1,36443149	
2	2	1,571429	3	-1,36443	1,7054108	-0,2477451	0,133982
3	1,5714286	1,705411	-1,3644315	-0,24775	1,7351358	0,029255402	0,029725
4	1,7054108	1,735136	-0,2477451	0,029255	1,7319964	-0,00051518	-0,00314
5	1,7351358	1,731996	0,0292554	-0,00052	1,7320507	-1,039E-06	5,43E-05

1.3) Secant method

- Example 2: Using Secant method, [0,1]

$$f(x) = 3x + \sin(x) - e^{x} \qquad \Delta_x = 1 \cdot 10^{-4}$$

n	x_0	x_1	$f(x_0)$	$f(x_1)$	x_2	$f(x_2)$	Δ_x
1	1	0	1,1231892	-1	0,4709896	0,265158816	
2	0	0,47099	-1	0,265159	0,3722771	0,029533669	-0,09871
3	0,4709896	0,372277	0,2651588	0,029534	0,3599042	-0,00129481	-0,01237
4	0,3722771	0,359904	0,0295337	-0,00129	0,3604239	5,53005E-06	0,00052
5	0,3599042	0,360424	-0,0012948	5,53E-06	0,3604217	1,02133E-09	-2,2E-06

1.4) Fixed point method

- The fixed point method is a root finding algorithm that seeks to find a fixed point of a function, which is a value that does not change when the function is applied to it. In other words, if we have a function f(x), the fixed point method seeks a value x* such that f(x*) = x*

$$f(x) = 0 \qquad x = g(x)$$

- If f(k) = 0, k become the root of f(x) and k =f(k)

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, 3, \ldots..$$

1.4) Fixed point method

- Example 1: Using fixed point method

$$f(x) = x^2 - 2x - 3 = 0$$

x=(2x+3)^1/2	x0			x=3/(x-2)	x0			(x2-3)/2	x0		
	4	3,316625			4	1,5			4	6,5	
	3,316625	3,103748			1,5	-6			6,5	19,625	
	3,103748	3,034385			-6	-0,375			19,625	191,0703	
	3,034385	3,01144			-0,375	-1,26316			191,0703	18252,43	
	3,01144	3,003811			-1,26316	-0,91935			18252,43	1,67E+08	
	3,003811	3,00127			-0,91935	-1,02762			1,67E+08	1,39E+16	
	3,00127	3,000423			-1,02762	-0,99088			1,39E+16	9,62E+31	
					-0,99088	-1,00305			9,62E+31	4,63E+63	
					-1,00305	-0,99898			4,63E+63	1,1E+127	
					-0,99898	-1,00034			1,1E+127	5,7E+253	diverges
					-1,00034	-0,99989					

1.4) Fixed point method

- Example 2: In 1225 Leonardo examined the root of the following equation: $f(x) = x^3 + 2x^2 + 10x - 20 = 0$

- Given that the converted g(x) $x_n = \dfrac{20}{x_{n-1}^2 + 2x_{n-1} + 10}$

is illustrated as:

Starting from x_0=1, how many times it was tried to find that the root is k=1,368808107.

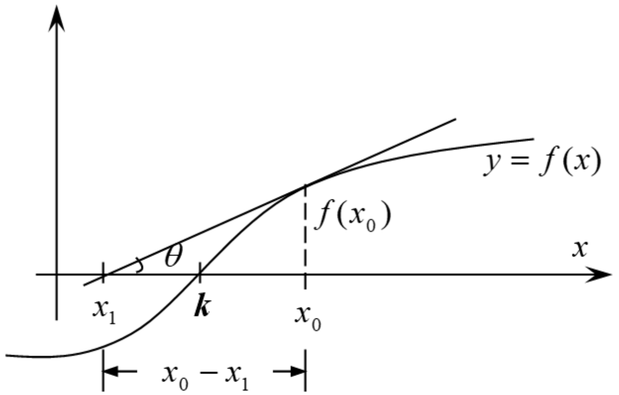
1,000000000	1,538461538	1,295019157	1,401825309	1,354209390	1,375298092	1,365929788	1,370086003
1,368241024	1,369059812	1,368696398	1,368857689	1,368786103	1,368817874	1,368803773	1,368810032
1,368807254	1,368808487	1,368807940	1,368808182	1,368808075	1,368808123	1,368808101	1,368808111
1,368808107	1,368808108	1,368808108					

1.5) Newton-Raphson method

- Perhaps the most widely used of all root-locating formulas is the Newton-Raphson equation

- The Newton-Raphson method is an iterative numerical method used for finding the roots of a function. It is particularly useful for finding the roots of nonlinear functions.

1.5) Newton-Raphson method



$$tg\theta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

- Rearrange this formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Called, Netwon-Rampson formula

1.5) Newton-Raphson method

- Common derivatives:

$$\frac{d}{dx}(x)=1$$
$$\frac{d}{dx}(\sin x)=\cos x$$
$$\frac{d}{dx}(\cos x)=-\sin x$$
$$\frac{d}{dx}(\tan x)=\sec^2 x$$
$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\csc x)=-\csc x\cot x$$
$$\frac{d}{dx}(\cot x)=-\csc^2 x$$
$$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x)=a^x\ln(a)$$
$$\frac{d}{dx}(\mathbf{e}^x)=\mathbf{e}^x$$
$$\frac{d}{dx}(\ln(x))=\frac{1}{x},\quad x>0$$
$$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\quad x\neq 0$$
$$\frac{d}{dx}(\log_a(x))=\frac{1}{x\ln a},\quad x>0$$

$$cosec\,\theta=\frac{1}{\sin\theta}$$
$$sec\,\theta=\frac{1}{\cos\theta}$$
$$\cot\theta=\frac{1}{\tan\theta}$$

$y = f(x) = h(x)\,.\,g(x)$ ise

$y' = f'(x) = h'(x)\,.\,g(x) + h(x)\,.\,g'(x)$

1.5) Newton-Raphson method

- Example 1: Using Newton method, find the root of the following equation between X_0=0

$$f(x) = 3x + \sin(x) - e^x$$

$$\Delta_x = 1 \cdot 10^{-6}$$

$$f(x) = 3x + \sin(x) - e^x \qquad f'(x) = 3 + \cos(x) - e^x$$

n	X0	F(x0)	F'(x0)	X1	Delta x
1	0,0000000	-1,0000000	3,0000000	0,3333333	
2	0,3333333	-0,0684177	2,5493445	0,3601707	0,026837380
3	0,3601707	-0,0006280	2,5022625	0,3604217	0,000250967
4	0,3604217	-0,0000001	2,5018142	0,3604217	0,000000022

1.5) Newton-Raphson method

- Example 2: Using Newton method, find the root of the following equation using $x_0 = 1.5$

$$\ln(x) = \sin^2(x) \qquad f'(x) = \frac{1}{x} - 2\sin x \cos x$$

n	X0	F(x0)	F'(x0)	X1	Delta x
1	1,500000000	-0,58953114	0,52554666	2,62174843	
2	2,621748432	0,71708590	1,24367132	2,04516048	1,12174843
3	2,045160482	-0,07588225	1,30163431	2,10345816	-0,57658795
4	2,103458157	0,00147221	1,35035330	2,10236792	0,05829767
5	2,102367919	0,00000044	1,34954192	2,10236759	-0,00109024
6	2,102367592	0,00000000	1,34954168	2,10236759	-0,00000033

1.5) Newton-Raphson method

- Example 3:

Problem Statement. Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$.

n	X0	F(x0)	F'(x0)	X1	Delta x
1	0	1	-2	0,5	
2	0,500000000	0,10653066	-1,60653066	0,566311003	0,50000000
3	0,566311003	0,00130451	-1,567615513	0,567143165	0,06631100
4	0,567143165	1,9648E-07	-1,567143362	0,56714329	0,00083216
5	0,567143290	4,44089E-15	-1,56714329	0,56714329	0,00000013
6	0,567143290	0	-1,56714329	0,56714329	0,00000000

2) Solution of linear systems

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

- where the a's are constant coefficients, the b's are constants, and n is the number of equations

2) Solution of linear systems

- Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

$$A = (a_{ij})_{mxn} \qquad 1 \le i \le m,\ 1 \le j \le n$$

$$A = (a_{ij})_{mxn} \qquad i = 1,2,\ldots\ldots,m, \quad j = 1,2,\ldots\ldots,n.$$

2) Solution of linear systems

- Matrix

$$A = B \quad \Leftrightarrow \quad a_{ij} = b_{ij}$$

$$C = A \pm B = a_{ij} \pm b_{ij}$$

$$A = A_{mxn} \qquad B = B_{nxp} \qquad C = AB \qquad C = C_{mxp}$$

$$kA = ka_{ij}$$

$$0 = (0)_{mxn}$$

$$I_n = (\delta_{ij})_{nxn} \qquad \delta_{ij} = \begin{cases} 1 & \mathrm{i} = \mathrm{j} \\ 0 & \mathrm{i} \neq \mathrm{j} \end{cases}$$

2) Solution of linear systems

- Square matrix (n=m)

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

- The diagonal consisting of the elements a_{11}, a_{22}, a_{33}, and a_{44} is termed the principal or main diagonal of the matrix

2) Solution of linear systems

- Identity matrix

$$[I] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Diagonal matrix

$$[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

2) Solution of linear systems

- Upper triangular matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix}$$

- Lower triangular matrix

$$[A] = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

2) Solution of linear systems

- Multiplication example

$$[Z] = [X][Y] = \begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}$$

- z_{11}=3x5+1x7=22
- z_{12}=3x9+1x2=29
- z_{21}=8x5+6x7=82
- z_{22}=8x9+6x2=84
- z_{31}=0x5+4x7=28
- z_{32}=0x9+4x2=8

$$[Z] = \begin{bmatrix} 22 & 29 \\ 82 & 84 \\ 28 & 8 \end{bmatrix}$$

2) Solution of linear systems

- Inverse of matrix

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\left|A^{-1}\right| = \frac{1}{|A|}$$

2) Solution of linear systems

- Inverse of matrix
- $\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^{-1} = ?$

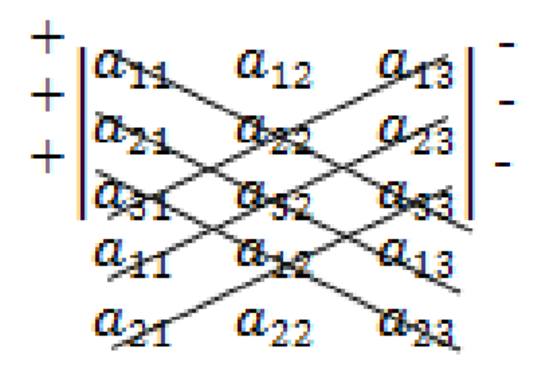
- Row operations

2) Solution of linear systems

- Determinant of matrix

$$|A| = |a_{11}| = a_{11}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$



$$|kB| = k|A| \qquad |A^T| = |A| \qquad |A^{-1}| = \frac{1}{|A|}$$

2.1) Cramer Method

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n = b_n
\end{aligned}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad x = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad b = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}
$$

2.1) Cramer Method

Det(A)

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

Cramer rule:

Replace b with the first column

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \; x_4 = \frac{\Delta_4}{\Delta}, \ldots\ldots\ldots, \quad x_n = \frac{\Delta_n}{\Delta}$$

2.1) Cramer Method

- Example 1: Solve the followings using Cramer rule

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= -1 \\
2x_1 - x_2 + 2x_3 &= -4 \\
4x_1 - x_2 + 4x_3 &= -2
\end{aligned}
$$

1		A				b			By hand			
		1	1	2		-1			1	1	2	
		2	-1	2		-4			2	-1	2	
		4	-1	4		-2		-8	4	-1	4	-4
								-2	1	1	2	-4
	Det(A)	2						8	2	-1	2	8
		-1	1	2						Det(A)	2	
		-4	-1	2								
		-2	-1	4								
	Det(A)	18				x1	9					
		1	-1	2								
		2	-4	2								
		4	-2	4								
	Det(A)	12				x2	6					
		1	1	-1								
		2	-1	-4								
		4	-1	-2								
	Det(A)	-16				x3	-8					

2.1) Cramer Method

- Example 2: Solve the followings using Cramer rule

$$2x_1 - x_2 + x_3 + 3x_4 = -1$$
$$x_1 + x_2 - x_3 - 4x_4 = 6$$
$$3x_1 - x_2 + x_3 + x_4 = 4$$
$$x_1 - 3x_2 + 3x_4 = -5$$

2		A					b			By hand				
		2	-1	1	3		-1			2	-1	1	3	
		1	1	-1	-4		6			1	1	-1	-4	
		3	-1	1	1		4			3	-1	1	1	
		1	3	0	3		-5			1	3	0	3	
	Det(A)	15									1	-1	-4	Det
									+	2	-1	1	1	9
		-1	-1	1	3						3	0	3	
		6	1	-1	-4									
		4	-1	1	1						1	-1	-4	Det
		-5	3	0	3				-	-1	3	1	1	15
											1	0	3	
	Det(A)	15					x1	1						
											1	1	-4	Det
		2	-1	1	3				+	1	3	-1	1	-54
		1	6	-1	-4						1	3	3	
		3	4	1	1									
		1	-5	0	3						1	1	-1	Det
									-	3	3	-1	1	-12
	Det(A)	-6,7E-15					x2	-4,4E-16			1	3	0	
		2	-1	-1	3									
		1	1	6	-4					**Det(A)**	**15**			
		3	-1	4	1									
		1	3	-5	3									
	Det(A)	45					x3	3						
		2	-1	1	-1									
		1	1	-1	6									
		3	-1	1	4									
		1	3	0	-5									
	Det(A)	-30					x4	-2						

2.2) Gauss Elimination !

- Transforming a matrix to a row echelon form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{array}\right]
\quad\Downarrow\quad
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
 & a'_{22} & a'_{23} & b'_2 \\
 & & a''_{33} & b''_3
\end{array}\right]
$$

Forward elimination

$$\Downarrow$$

$$
\begin{aligned}
x_3 &= b''_3 / a''_{33} \\
x_2 &= (b'_2 - a'_{23}x_3)/a'_{22} \\
x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}
\end{aligned}
$$

Back substitution

2.2) Gauss Elimination

- Example 1: Use gauss elimination method

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &= 5, \\
2x_1 + 5x_2 + x_3 &= -3, \\
2x_1 + x_2 + 3x_3 &= 11
\end{aligned}
$$

1							
	R1	3	2	1	5		
	R2	2	5	1	-3		
	R3	2	1	3	11		
	R1/3	1	0,666667	0,333333	1,666667		
	R2	2	5	1	-3		
	R3	2	1	3	11		
	R1	1	0,666667	0,333333	1,666667		
	R2-2R1	0	3,666667	0,333333	-6,33333		
	R3-2R1	0	-0,33333	2,333333	7,666667		
	R1	1	0,666667	0,333333	1,666667		
	R2/3,666667	0	1	0,090909	-1,72727		
	R3	0	-0,33333	2,333333	7,666667		
	R1	1	0,666667	0,333333	1,666667	x1	2
	R2	0	1	0,090909	-1,72727	x2	-2
	R3+0,33333*R2	0	0	2,363636	7,090909	x3	3

2.2) Gauss Elimination

- Example 2: Use gauss elimination method

- x + y – z = -2
- 2x – y + z = 5
- -x + 2y + 2z = 1

2

R1	1	1	-1	-2			
R2	2	-1	1	5			
R3	-1	2	2	1			
R1	1	1	-1	-2			
R2-2R1	0	-3	3	9			
R3+R1	0	3	1	-1			
R1	1	1	-1	-2			
R2	0	-3	3	9			
R3+R2	0	0	4	8			
R1	1	1	-1	-2			
R2/-3	0	1	-1	-3			
R3/4	0	0	1	2			
R1	x	y	(-z)	-2		x	1
R2		y	(-z)	-3		y	-1
R3			z	2		z	2

2.3) Jacobi method

- Each unknown is placed alone at the left side of the equation

$$x_i = \frac{1}{a_{i,j}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j \right], \qquad i = 1, 2, \ldots\ldots\ldots, n$$

- Start finding the solution by using initial values (i.e., 0, 0, 0) and then continue iteratively

$$x_i^{(k+1)} = \frac{1}{a_{i,j}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)} \right], \qquad i = 1, 2, \ldots\ldots\ldots, n$$

- **Convert the order to make diagonals dominant!!**

2.3) Jacobi method

- Example 1: Use Jacobi method by using (0,0,0) initial values

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ x_1 + 2x_2 - 5x_3 &= -1 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \end{aligned}$$

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - 5x_3 &= -1 \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{6}[11 + 2x_2 - x_3], \\ x_2 &= \frac{1}{7}[5 + 2x_1 - 2x_3], \\ x_3 &= \frac{1}{5}[1 + x_1 + 2x_2], \end{aligned}$$

1

n	x1	x2	x3
1	0	0	0
2	1,833333	0,714286	0,2
3	2,038095	1,180952	0,852381
4	2,084921	1,053061	1,08
5	2,004354	1,001406	1,038209
6	1,994101	0,990327	1,001433
7	1,996537	0,997905	0,994951
8	2,000143	1,000453	0,998469
9	2,000406	1,000478	1,00021
10	2,000124	1,000056	1,000273
11	1,999973	0,999958	1,000047

2.3) Jacobi method

- Example 2: Use Jacobi method by using (1,1,1) initial values

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

2

	k	0	x1	x2	x3
x1	0,333333	7,85		0,1	0,2
x2	0,142857	-19,3	-0,1		0,3
x3	0,1	71,4	-0,3	0,2	

n	x1	x2	x3
1	1	1	1
2	2,716667	-2,72857	7,13
3	3,001048	-2,49038	7,003929
4	3,000583	-2,49985	7,000161
5	3,000016	-2,5	6,999986
6	2,999999	-2,5	6,999999
7	3	-2,5	7
8	3	-2,5	7
9	3	-2,5	7
10	3	-2,5	7
11	3	-2,5	7

2.4) Gauss-Siedel

- Similar to Jacobi method but show difference in terms of iteration
- A special case of Jacobi
- When we find x_1, this is used exactly for the following variable (i.e., x_2)
- **Note: Diagonal should be in its dominant form!**

2.4) Gauss-Siedel

- Example: : Use Gauss-Siedel method by using (0,0,0) initial values

$$6x_1 - 2x_2 + x_3 = 11$$
$$x_1 + 2x_2 - 5x_3 = -1$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

$$x_1 = \frac{1}{6}[11 + 2x_2 - x_3],$$
$$x_2 = \frac{1}{7}[5 + 2x_1 - 2x_3],$$
$$x_3 = \frac{1}{5}[1 + x_1 + 2x_2],$$

1

n	x1	x2	x3	Error 1	Error 2	Error 3	Error
1	0	0	0				
2	1,833333	1,238095	1,061905				
3	2,069048	0,934694	1,061905				
4	1,967914	1,002041	0,987687				
5	2,002732	0,994351	0,994399				
6	1,99905	1,002381	0,998287	0,003682	0,00803	0,003888	0,009652
7	2,001079	1,000218	1,000762	0,002029	0,002163	0,002476	0,003863
8	1,999946	1,000091	1,000303	0,001134	0,000128	0,000459	0,00123
9	1,99998	0,999898	1,000025	3,4E-05	0,000193	0,000278	0,00034
10	1,999962	0,999987	0,999955	1,79E-05	8,91E-05	7,03E-05	0,000115
11	2,000003	1,000002	0,999987	4,14E-05	1,5E-05	3,2E-05	5,45E-05

2.4) Gauss-Siedel

- Example: : Use Gauss-siedel by using (1,1,1) initial values

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

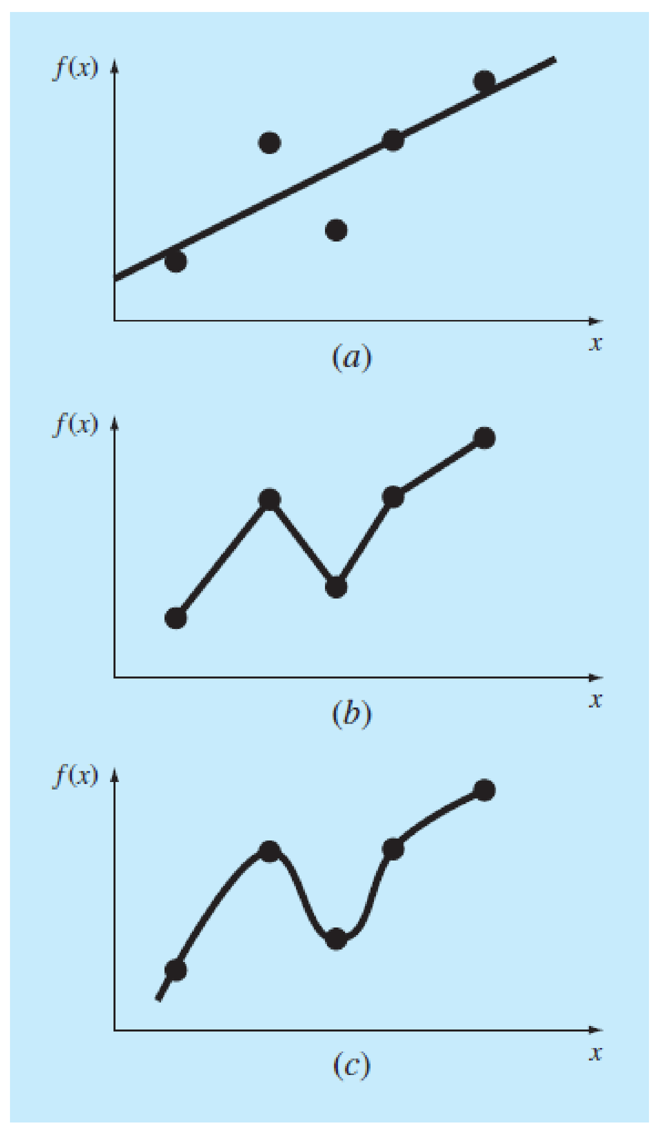
2

	k	0	x1	x2	x3
x1	0,333333	7,85		0,1	0,2
x2	0,142857	-19,3	-0,1		0,3
x3	0,1	71,4	-0,3	0,2	

n	x1	x2	x3
1	1	1	1
2	2,716667	-2,7531	7,003438
3	2,991793	-2,49974	7,000252
4	3,000026	-2,49999	6,999999
5	3	-2,5	7
6	3	-2,5	7
7	3	-2,5	7
8	3	-2,5	7
9	3	-2,5	7
10	3	-2,5	7
11	3	-2,5	7

3) Interpolation (Curve fitting)

- Data are often given for discrete values along a continuum. However, you may require estimates at points between the discrete values.
- In addition, you may require a simplified version of a complicated function. One way to do this is to compute values of the function at a number of discrete values along the range of interest. Then, a simpler function may be derived to it these values. Both of these applications are known as curve fitting.
- a) lest squares regression, (b) linear interpolation, (c) curvilinear interpolation



3.1) Lagrange Polynomial

- While the exact values are obtained at the points given in the Lagrangian Polynomial, approximate results are obtained at the intermediate values.
- As in all polynomial fittings, the degree of the polynomial must be at least 1 order less than the given number of points. For instance, the polynomial to be fitted for 4 f_n values corresponding to the 4 X_n points can be maximum 3rd order. Lagrange approximation polynomial:

$$P(x) = L_0(x) f_0 + L_1(x) f_1 + \ldots\ldots + L_i(x) f_i + \ldots\ldots + L_n(x) f_n$$

$$L_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

3.1) Lagrange Polynomial

- Example 1: Find a 3rd order polynomial using Lagrange Polynomials with the following values obtained during an experiment.

	0	**1**	**2**	**3**	**4**
x	3.2	2.7	1.0	4.8	5.6
$f(x)$	22.0	17.8	14.2	38.3	51.7

$$P_3(x) = \frac{(x-2.7)(x-1.0)(x-4.8)}{(3.2-2.7)(3.2-1.0)(3.2-4.8)}(22.0) + \frac{(x-3.2)(x-1.0)(x-4.8)}{(2.7-3.2)(2.7-1.0)(2.7-4.8)}(17.8) +$$
$$\frac{(x-3.2)(x-2.7)(x-4.8)}{(1.0-3.2)(1.0-2.7)(1.0-4.8)}(14.2) + \frac{(x-3.2)(x-2.7)(x-1.0)}{(4.8-3.2)(4.8-2.7)(4.8-1.0)}(38.3)$$

3.1) Lagrange Polynomial

- Example 2: According to the provided a and cos(a) values find cos($8^0 40'$) use Lagrange third order polynomial.

α	cosα
8^0	0.99027
$8^0 30'$	0.98902
9^0	0.98769
$9^0 30'$	0.98629

$$P_3(x)=\frac{(x-8^\circ.30')(x-9^\circ)(x-9^\circ30')}{(8^\circ-8^\circ.30')(8^\circ-9^\circ)(8^\circ-9^\circ30')}(0.99027)+\frac{(x-8^\circ)(x-9^\circ)(x-9^\circ30')}{(8^\circ.30'-8^\circ)(8^\circ.30'-9^\circ)(8^\circ.30'-9^\circ30')}(0.98902)+$$
$$\frac{(x-8^\circ)(x-8^\circ.30')(x-9^\circ30')}{(9^\circ-8^\circ)(9^\circ-8^\circ.30')(9^\circ-9^\circ30')}(0.98769)+\frac{(x-8^\circ)(x-8^\circ.30')(x-9^\circ)}{(9^\circ30'-8^\circ)(9^\circ30'-8^\circ.30')(9^\circ30'-9^\circ)}(0.98629)$$

$$P_3(8^\circ40')=\frac{(8^\circ40'-8^\circ.30')(8^\circ40'-9^\circ)(8^\circ40'-9^\circ30')}{(8^\circ-8^\circ.30')(8^\circ-9^\circ)(8^\circ-9^\circ30')}(0.99027)+$$
$$=\frac{(8^\circ40'-8^\circ)(8^\circ40'-9^\circ)(8^\circ40'-9^\circ30')}{(8^\circ.30'-8^\circ)(8^\circ.30'-9^\circ)(8^\circ.30'-9^\circ30')}(0.98902)+$$
$$=\frac{(8^\circ40'-8^\circ)(8^\circ40'-8^\circ.30')(8^\circ40'-9^\circ30')}{(9^\circ-8^\circ)(9^\circ-8^\circ.30')(9^\circ-9^\circ30')}(0.98769)+$$
$$=\frac{(8^\circ40'-8^\circ)(8^\circ40'-8^\circ.30')(8^\circ40'-9^\circ)}{(9^\circ30'-8^\circ)(9^\circ30'-8^\circ.30')(9^\circ30'-9^\circ)}(0.98629)=0.98859$$

- İf 0.5xa = 30’ → 8.40’ = 8.6667
- Consider 8, 8.5, 9, 9.5 and find for 8.667

3.2) Newton Polynomial

- In Newton's Polynomial, a polynomial is obtained by using linearapproximation to numerical values at a given point

$$P(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + (x - x_0)(x - x_1)(x - x_2)a_3 + \ldots..$$
$$\ldots. + (x - x_0)\ldots..(x - x_{n-1})a_n$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1], \quad \text{ve} \quad \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = f[x_i, x_{i+1}]$$

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2] \quad \text{ve} \quad \frac{f[x_{i+1} - x_{i+2}] - f[x_i - x_{i+1}]}{x_{i+2} - x_i} = f[x_i, x_{i+1}, x_{i+2}]$$

$$a_n = \frac{f[x_1, x_2, \ldots\ldots., x_n] - f[x_0, x_1, \ldots\ldots., x_{n-1}]}{x_n - x_0}$$

3.2) Newton Polynomial

- In Table format

$(x_{i+3}-x_i)$	$(x_{i+2}-x_i)$	$(x_{i+1}-x_i)$	x_i	f_i	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
			x_0	f_0			
		(x_1-x_0)			$f[x_0,x_1]$		
	(x_2-x_0)		x_1	f_1		$f[x_0,x_1,x_2]$	
(x_3-x_0)		(x_2-x_1)			$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$
			x_2	f_2		$f[x_1,x_2,x_3]$	
	(x_3-x_1)	(x_3-x_2)			$f[x_2,x_3]$		
			x_3	f_3			

3.2) Newton Polynomial

- Example 1: Find a polynomial using Newton interpolation method based on the provided information.

x_i	5	3	4	1
y_i	53	19	30	9

$(x_{i+3}-x_i)$	$(x_{i+2}-x_i)$	$(x_{i+1}-x_i)$	x_i	f_i	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
			5	**53**			
		-2			**17**		
	-1		3	19		**6**	
-4		1			11		**1**
	-2		4	30		2	
		-3			7		
			1	9			

$$P(x) = 53 + 17(x-5) + 6(x-5)(x-3) + 1(x-5)(x-3)(x-4)$$

3.2) Newton Polynomial

- Example 2: Find a polynomial using Newton interpolation method based on the provided information. Using this polynomial, estimate the value of y(6).

x_i	2	4	7	9
y_i	2.5	4.0	3.4	2.4

$(x_{i+3}-x_i)$	$(x_{i+2}-x_i)$	$(x_{i+1}-x_i)$	x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
			2	**2.5**			
		2			**0.75**		
	5		4	4.0		**-0.19**	
7		3			-0.2		**0.01857**
	5		7	3.4		-0.06	
		2			-0.5		
			9	2.4			

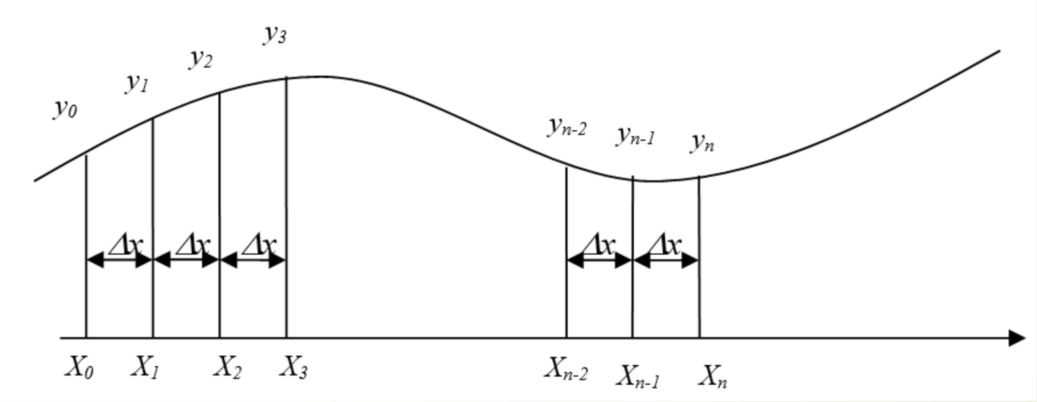
$$P(x) = 2.5 + 0.75(x-2) - 0.19(x-2)(x-4) + 0.01857(x-2)(x-4)(x-7)$$

$$P(6) = 2.5 + 0.75(6-2) - 0.19(6-2)(6-4) + 0.01857(6-2)(6-4)(6-7) = 3.83144$$

3.3) Finite Difference Polynomial

- Finite Difference Method is actually a special case of Newton Interpolation Polynomial. In this method, unlike the Newton interpolation polynomial, the intervals of the data are equal to each other.

$$
\begin{aligned}
x_1 &= x_0 + \Delta x & & & x_1 &= x_0 + \Delta x \\
x_2 &= x_1 + \Delta x & & & x_2 &= x_0 + 2\Delta x \\
x_3 &= x_2 + \Delta x & \Rightarrow & & x_3 &= x_0 + 3\Delta x \\
&\vdots & & & &\vdots \\
x_n &= x_{n-1} + \Delta x & & & x_n &= x_0 + n\Delta x
\end{aligned}
$$



3.3) Finite Difference Polynomial

- Similar to Newton Interpolation Polynomial:

$$a_1 = f\left[x_0, x_1\right] = \frac{f_1 - f_2}{x_1 - x_0} = \frac{\Delta f_0}{\Delta x} \qquad \text{ve} \qquad f\left[x_i, x_{i+1}\right] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = \frac{\Delta fi}{\Delta x}$$

$$a_2 = f\left[x_0, x_1, x_2\right] = \frac{f\left[x_1, x_2\right] - f\left[x_0, x_1\right]}{x_2 - x_0} = \frac{\frac{\Delta f_1}{\Delta x} - \frac{\Delta f_0}{\Delta x}}{2\Delta x} = \frac{1}{2!}\frac{\Delta f_1 - \Delta f_0}{(\Delta x)^2} = \frac{1}{2!}\frac{\Delta^2 f_0}{(\Delta x)^2}$$

$$f\left[x_i, x_{i+1}, x_{i+2}\right] = \frac{f\left[x_{i+1}, x_{i+2}\right] - f\left[x_i, x_{i+1}\right]}{x_{i+2} - x_i} = \frac{\frac{\Delta f_{i+1}}{\Delta x} - \frac{\Delta f_i}{\Delta x}}{2\Delta x} = \frac{1}{2!}\frac{\Delta f_{i+1} - \Delta f_0}{(\Delta x)^2} = \frac{1}{2!}\frac{\Delta^2 f_i}{(\Delta x)^2}$$

$$\frac{f\left[x_1, x_2, \ldots\ldots, x_n\right] - f\left[x_0, x_1, \ldots\ldots, x_n\right]}{x_n - x_0} = \frac{1}{n!}\frac{\Delta^n f_0}{(\Delta x)^n}$$

3.3) Finite Difference Polynomial

- Example 1: Obtain a polynomial using finite difference interpolation with the following data. Using this polynomial, find an approximate value for x=0.54.

x_i	0.4	0.5	0.6	0.7
$y_i = e^{x_i}$	1.4918	1.6487	1.8221	2.0138

			x_i	y_i	Δf	$\Delta^2 f$	$\Delta^3 f$	
		x0	0.4	**1.491825**				(f0)
Delta x	0.1				**0.156897**			
x	0.54	x1	0.5	1.648721		**0.016501**		(f1)
					0.173398		**0.001735**	
		x2	0.6	1.822119		**0.018236**		(f2)
					0.191634			
		x3	0.7	2.013753				(f3)
			n					
			0	a0	1.491825			
			1	a1	1.568966			
			2	a2	0.825048		Eğri	1.7160
			3	a3	0.289237			

$$P(x) = 14918 + 1.5690(x - 0.4) + 0.825(x - 0.4)(x - 0.5) + 0.3(x - 0.4)(x - 0.5)(x - 0.6)$$

$$P(x) = 14918 + 1.5690(0.14) + 0.825(0.14)(0.04) + 0.3(0.14)(0.04)(-0.06) = 1.7159792$$

$$y = e^{0.54} = 1.716006862 \quad \Rightarrow \Delta y = 2.8 \cdot 10^{-5} \quad \therefore$$

$y_i = e^{x_i}$

3.3) Finite Difference Polynomial

- Example 2: Obtain a polynomial using finite difference interpolation with the following data. Using this polynomial, find an approximate value for x=0.08.

x_i	0.0585	0.0770	0.0955	0.1140
y_i	0.14201	0.17776	0.20950	0.23827

				x_i	y_i	Δf	$\Delta^2 f$	$\Delta^3 f$	
		Check	x0	0.0585	**0.14201**				(f0)
Delta x	0.0185	0.0185				**0.03575**			
x	0.08		x1	0.077	0.17776		**-0.00401**		(f1)
						0.03174		**0.00104**	
			x2	0.0955	0.2095		**-0.00297**		(f2)
						0.02877			
			x3	0.114	0.23827				(f3)
				n					
				0	a0	0.14201			
				1	a1	1.932432			
				2	a2	-5.85829		Eğri	0.183152
				3	a3	27.37581			

$$P(x) = 0.14201 + 1.932432(x-0.0585) - 5.858291(x-0.0585)(x-0.0077) + 27.375805(x-0.0585)(x-0.077)(x-0.0955)$$

$$P(0.08) = 0.14201 + 1.932432(0.0215) - 5.858291(0.0215)(0.003) + 27.375805(0.0215)(0.003)(-0.0155) = 0.183152 \quad \therefore$$

3.4) Least Squares

- The Least Squares Method is one of the most widely used interpolation methods. The reason why it is so preferred is that it is not only easy to remember, but also very good in precision. However, it is not preferred to obtain polynomials larger than the 4th or 5th order because of the difficulty and the low sensitivity of the approximate results.
- Finding the difference between Y and the polynomial:

$$y(x_i) = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots + a_n x^n$$

$$\varphi = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} \left[y(x_i) - Y_i \right]^2 = \sum_{i=1}^{N} \left[a_0 + a_1 x_i + a_2 x_i^2 + \dots\dots\dots + a_n x_i^n - Y_i \right]^2$$

3.4) Least Squares

- Making the minimum (least) of this, the derivative shoudl be 0: $\frac{\partial \varphi}{\partial a_i} = 0 \text{ ve } \frac{\partial^2 \varphi}{\partial a_i^{\,2}} > 0$

- If we apply this for each a:

$$\frac{\partial \varphi}{\partial a_0} = \sum_{i=1}^{N} 2\left[a_0 + a_1 x_i + a_2 x_i^{\,2} + \dots\dots\dots + a_n x_i^{\,n} - Y_i\right](1) = 0$$

$$\frac{\partial \varphi}{\partial a_1} = \sum_{i=1}^{N} 2\left[a_0 + a_1 x_i + a_2 x_i^{\,2} + \dots\dots\dots + a_n x_i^{\,n} - Y_i\right](x_i) = 0$$

$$\frac{\partial \varphi}{\partial a_2} = \sum_{i=1}^{N} 2\left[a_0 + a_1 x_i + a_2 x_i^{\,2} + \dots\dots\dots + a_n x_i^{\,n} - Y_i\right](x_i^{\,2}) = 0$$

$$\vdots$$

$$\vdots$$

$$\frac{\partial \varphi}{\partial a_n} = \sum_{i=1}^{N} 2\left[a_0 + a_1 x_i + a_2 x_i^{\,2} + \dots\dots\dots + a_n x_i^{\,n} - Y_i\right](x_i^{\,n}) = 0$$

If we refine them:

$$a_0 N + a_1 \sum_{i=1}^{N} x_i + a_2 \sum_{i=1}^{N} x_i^{\,2} + \dots\dots\dots + a_n \sum_{i=1}^{N} x_i^{\,n} = \sum_{i=1}^{N} Y_i$$

$$a_0 \sum_{i=1}^{N} x_i + a_1 \sum_{i=1}^{N} x_i^{\,2} + a_2 \sum_{i=1}^{N} x_i^{\,3} + \dots\dots\dots + a_n \sum_{i=1}^{N} x_i^{\,n+1} = \sum_{i=1}^{N} Y_i x_i$$

$$\vdots$$

$$\vdots$$

$$a_0 \sum_{i=1}^{N} x_i^{\,n} + a_1 \sum_{i=1}^{N} x_i^{\,n+1} + a_2 \sum_{i=1}^{N} x_i^{\,n+3} + \dots\dots\dots + a_n \sum_{i=1}^{N} x_i^{\,2n} = \sum_{i=1}^{N} Y_i x_i^{\,n}$$

3.4) Least Squares

- The matrix form can be generated:

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \cdots & \sum_{i=1}^{N} x_i^n \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \cdots & \sum_{i=1}^{N} x_i^{n+1} \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^4 & \cdots & \sum_{i=1}^{N} x_i^{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{N} x_i^n & \sum_{i=1}^{N} x_i^{n+1} & \sum_{i=1}^{N} x_i^{n+2} & \cdots & \sum_{i=1}^{N} x_i^{2n} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{N} Y_i \\ \sum_{i=1}^{N} Y_i x_i \\ \sum_{i=1}^{N} Y_i x_i^2 \\ \vdots \\ \vdots \\ \sum_{i=1}^{N} Y_i x_i^n \end{Bmatrix} \Rightarrow \boldsymbol{Ba=C}$$

- B is the coefficient matrix and it can be expressed as: $\boldsymbol{B=A.A^T}$
- Here, A is regulation matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_i & \cdots & x_n \\ x_1^{\ 2} & x_2^{\ 2} & x_3^{\ 2} & & x_i^{\ 2} & & x_n^{\ 2} \\ x_1^{\ 3} & x_2^{\ 3} & x_3^{\ 3} & & x_i^{\ 3} & & x_n^{\ 3} \\ & & & & & & \\ x_1^{\ N} & x_2^{\ N} & x_3^{\ N} & \cdots & x_i^{\ N} & \cdots & x_n^{\ N} \end{bmatrix}$$

3.4) Least Squares

- Example 1: Using the data in the table below, derive a quadratic (second order) polynomial with respect to x by Least Squares Method

x_i	0.0500	0.1100	0.1500	0.3100	0.4600	0.5200	0.7000	0.7400	0.8200	0.9800	1.1700
y_i	0.9560	0.8900	0.8320	0.7170	0.5710	0.5390	0.3780	0.3700	0.3060	0.2420	0.1040

	1	1	1	1	1	1	1	1	1	1	1
A	0.05	0.11	0.15	0.31	0.46	0.52	0.7	0.74	0.82	0.98	1.17
	0.0025	0.0121	0.0225	0.0961	0.2116	0.2704	0.49	0.5476	0.6724	0.9604	1.3689

	1	0.05	0.0025
	1	0.11	0.0121
	1	0.15	0.0225
	1	0.31	0.0961
	1	0.46	0.2116
A(T)	1	0.52	0.2704
	1	0.7	0.49
	1	0.74	0.5476
	1	0.82	0.6724
	1	0.98	0.9604
	1	1.17	1.3689

	11	6.01	4.6545	0.595502	5.905		5.905
B	6.01	4.6545	4.114963	9.849035	5.905	C	2.18387
	4.6545	4.114963	3.916128	6.935848	5.905		1.335721

	0.595502	-2.01591	1.410479		0.9980
B(-1)	-2.01591	9.849035	-7.95311	a	-1.0180
	1.410479	-7.95311	6.935848		0.2247

	C	
5.905	2.18387	1.335721

$$\left\{ \begin{array}{c} \sum_{i=1}^{N} Y_i \\ \sum_{i=1}^{N} Y_i x_i \\ \sum_{i=1}^{N} Y_i x_i^2 \\ \vdots \\ \vdots \\ \sum_{i=1}^{N} Y_i x_i^n \end{array} \right\}$$

$$y = 0.998 - 1.018x + 0.2247x^2 \quad \therefore$$