GENERAL RELATIVITY
FINAL EXAM

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Student's Name:

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1. a) Write down the line element (the metric) for the empty space around a spherical object of mass M in the weak-field limit.

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1+\frac{2GM}{c^2 r}\right)\left(dx^2+dy^2+dz^2\right)$$

b) How can the metric tensor $g_{\mu\nu}$ be written in the weak-field limit in (nearly) Cartesian coordinates? Write down the matrices for the two terms. for the Sch. solution

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad h_{\mu\nu} = \begin{pmatrix} \frac{r_s}{r} & 0 & 0 & 0 \\ 0 & \frac{r_s}{r} & 0 & 0 \\ 0 & 0 & \frac{r_s}{r} & 0 \\ 0 & 0 & 0 & r_s/r \end{pmatrix} ; \quad r_s = \frac{2GM}{c^2}$$

c) $g^{\mu\nu}$ can be written as $g^{\mu\nu} = a^{\mu\nu} + b^{\mu\nu}$.
What are $a^{\mu\nu}$ and $b^{\mu\nu}$?

$$a^{\mu\nu} = \eta^{\mu\nu}, \quad b^{\mu\nu} = -h^{\mu\nu}$$

d) Is there an event horizon in the weak-field limit? Explain.

There is no event horizon in the weak-field limit because $g_{xx} = g_{yy} = g_{zz} = (1 + r_s/r)$ does not become zero for any value of r.

2. For a binary star system with masses m_1 and m_2 separated by a distance D rotating around their center of mass with angular frequency ω, the metric perturbation is given by

$$h_{TT}^{jk} = -\frac{4GM\eta D^2\omega^2}{c^4 R_0}\begin{bmatrix} X & \sin\left[2\omega\left(t-\frac{R_0}{c}\right)\right] & 0 \\ Y & -\cos\left[2\omega\left(t-\frac{R_0}{c}\right)\right] & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

, where $M \equiv m_1+m_2$, $\eta \equiv \frac{m_1 m_2}{M^2}$, and R_0 is the distance in the z direction from the center of mass of the system.

a) What is X equal to, and why?

$X = \cos[2\omega(t-R_0/c)]$ because h_{TT}^{jk} is traceless.

b) What is Y equal to?

$$Y = \sin[2\omega(t-R_0/c)]$$

c) Write h_{TT}^{jk} as a sum of "plus" and "cross" polarization waves.

$$h_{TT}^{jk} = -\overbrace{\left(\frac{4GM\eta D^2\omega^2}{c^4 R_0}\right)}^{A_+}\cos[2\omega t\ ]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ \ \end{matrix}$$

$$-\underbrace{\left(\frac{4GM\eta D^2\omega^2}{c^4 R_0}\right)}_{A_\otimes}\sin[\quad]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) What is the angular frequency of these gravitational waves?

$$\omega_{GW} = 2\omega$$

e) What are the amplitudes A_+ and $A_\otimes$?

$$A_+ = A_\otimes = \frac{4GM\eta D^2 \omega^2}{c^4 R_0}$$

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3. In question 2, the binary star system is in the xy-plane and rotates around the z-axis. Now consider a system in the yz-plane rotating around the x-axis.

a) Write down h^{jk}_{TT} for this system.

$$h^{jk}_{TT} = -\frac{4GM\eta D^2 \omega^2}{c^4 R_0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos[2\omega(\quad)], & \sin[2\omega(\quad)] \\ 0 & \sin[2\omega(\quad)], & -\cos[2\omega(\quad)] \end{bmatrix}$$

b) What is the type of polarization of these gravitational waves?

Circular polarization

4. In the weak-field limit the metric tensor can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. The so-called trace-reversed metric perturbation $H_{\mu\nu}$ is defined as $H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$, where $h \equiv h^{\mu}{}_{\mu}$.

a) What is the Lorentz gauge condition on $H^{\mu\nu}$?

$$\partial_\mu H^{\mu\nu} = 0$$

b) Write down the weak-field Einstein Equation in the presence of matter and energy.

$$\Box^2 H^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}, \text{ where } \Box^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

c) Write down the same equation in empty space.

$$\Box^2 H^{\mu\nu} = 0$$

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5. a) Write down Maxwell's equations in Electromagnetism

$$\vec{\nabla}\cdot\vec{E} = 4\pi k_e \rho_Q$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

b) Write down the corresponding equations in Gravitation.

$$\vec{\nabla}\cdot\vec{E}_G = -4\pi G\rho_M, \quad \vec{\nabla}\times\vec{E}_G = -\frac{\partial\vec{B}_G}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}_G = 0 \quad\quad \vec{\nabla}\times\vec{B}_G = -\frac{4\pi G}{c^2}\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}_G}{\partial t}$$

c) What is the "gravitoelectric" field?

It's the gravitational field $\vec{g}$

d) Write down the Lorentz force equation for a particle of mass m, velocity $\vec{v}$ moving in a superimposed gravitoelectric and gravitomagnetic fields.

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} = m\left(\vec{E}_G + \vec{v} \times 4\vec{B}_G\right)$$

e) What is the Lense-Thirring Precession? Describe briefly

A gyroscope in the equatorial plane of the Earth precesses due to the gravitomagnetic field at the position of the gyroscope created by the Earth.

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6. The Reissner-Nördström metric describes the spacetime geometry outside a spherical object with mass M and electric charge Q. It is given by (in GR units)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

6a.) Calculate the radii of the infinite red shift surfaces.

$$r^2 - 2Mr + Q^2 = 0$$

$$r_{1,2} = \frac{2M \pm \sqrt{4M^2 - 4Q^2}}{2}$$

$$= M \pm \sqrt{M^2 - Q^2}$$

$$r_+ = M + \sqrt{M^2 - Q^2}$$

$$r_- = M - \sqrt{M^2 - Q^2}$$

b) What are the radii of the event horizons?

same as r_+ and r_-

c) What is the condition for a naked singularity?

For a naked singularity $r_\pm$ become imaginary. This occurs when $Q > M$

d) For what value of Q the radius of the event horizon is equal to the radius of the Schwarzschild event horizon?

For $Q = 0 \Rightarrow$ $r_+ = M + M = 2M = r_S$

$r_- = M - M = 0$

e) What does cosmic censorship require Q^2 to be?

For $r_\pm$ to be real Q must be less than M.

7. In an alternative theory, the metric outside a spherical object with mass M and electric charge $Q>0$ is given by

$$ds^2 = -\left(1-2\frac{GM}{c^2 r}+2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)c^2 dt^2$$

$$+\left(1-\frac{2GM}{c^2 r}+2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

a) What is the infinitesimal radial distance along a radial line?

$$ds = dR = \left(1-\frac{2GM}{c^2 r}+\frac{2e}{m_e}\frac{k_e Q}{c^2 r}\right)^{-1/2} dr$$

b) Calculate the radial distance between two events A and B with coordinates r_A and r_B along such a line in the limit $\frac{2GM}{c^2 r} \ll 1$ and $\frac{e}{m_e}\frac{k_e Q}{c^2 r} \ll 1$.

$$R = \int dR = \int_{r_A}^{r_B} \left(1-\frac{2GM}{c^2 r}+\frac{2e}{m_e}\frac{k_e Q}{c^2 r}\right)^{-1/2} dr \approx \int_{r_A}^{r_B} \left(1+\frac{GM}{c^2 r}-\frac{e k_e Q}{m_e c^2 r}\right) dr$$

$$R \approx r_B - r_A + \left(\frac{GM}{c^2} - \frac{e}{m_e}\frac{k_e Q}{c^2}\right) \ln (r_B - r_A)$$

8. Consider a hypothetical two-dimensional space-time metric given by

$$ds^2 = -c^2 dt^2 + 4c\,dt\,dr + \left(1-\frac{R_0}{r}\right)^{-1} dr^2$$

a) Write down the matrix for the metric $g_{\mu\nu}$ $(\mu=0,1)$.

$$g_{\mu\nu} = \begin{pmatrix} -1 & 2 \\ 2 & \left(1-\frac{R_0}{r}\right)^{-1} \end{pmatrix}$$

b) Find the matrix for $g^{\mu\nu}$

Let the inverse matrix be $g^{\mu\nu} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} -1 & 2 \\ 2 & A \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -a+2c, & -b+2d \\ 2a+Ac, & 2b+Ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$-a+2c = 1$ ①

$-b+2d = 0$ ② $\Rightarrow \boxed{d = \frac{b}{2}}$

$2a+Ac = 0$ ③

$2b+Ad = 1$ ④ $\Rightarrow 2b + A\frac{b}{2} = 1$

$b\left(2+\frac{A}{2}\right) = 1$

$\boxed{b = \frac{1}{2+\frac{A}{2}}}$

$\boxed{b = \frac{2}{4+A}}$

③ $\Rightarrow c = -\frac{2a}{A}$

① $\Rightarrow -a + \frac{4a}{A} = 1$

$a\left(-1+\frac{4}{A}\right) = 1$

$\boxed{a = \frac{1}{-1+\frac{4}{A}}}$

~~① + ② $\Rightarrow +(a+b) = 2(c+d)$~~

~~③ + ④ $\Rightarrow 2(a+b) = -A(c+d)$~~

~~$4(c+d) = -A(c+d)$~~

$\Rightarrow$ $\cancel{A=-4}$

$\boxed{d = \frac{1}{4+A}}$

$\boxed{a = \frac{A}{4-A}}$ $\boxed{c = -\frac{2}{4-A}}$

9. a) Consider the metric in question 7 again.

Find Q in terms of M and other parameters such that the proper time of the test particle is equal to the coordinate time.

$$d\tau = dt \Rightarrow \left(1 - \frac{2GM}{c^2 r} + 2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right) = 1$$

$$\Rightarrow -GM + \frac{e}{m_e} k_e Q = 0 \Rightarrow Q = \frac{m_e}{e}\frac{G}{k_e} M$$

b) Find the condition on R so that this spherical object becomes a black hole, where R is the radius of the object.

$$r - \frac{2GM}{c^2} + 2\frac{e}{m_e}\frac{k_e Q}{c^2} = 0$$

$$r_{EH} = \frac{2GM}{c^2} - \frac{2e}{m_e}\frac{k_e Q}{c^2}$$

For $R < r_{EH}$ the object becomes a black hole.

c) For laboratory-size spheres we have $\frac{e}{m_e}\frac{k_e Q}{c^2} \gg \frac{GM}{c^2}$.

Can such a sphere become a black hole for positrons? Explain why?

In this case, $r_{EH} \approx -\frac{2e}{m_e}\frac{k_e Q}{c^2}$, which is negative.

So, such a sphere cannot be a black hole.

10. Consider a hypothetical spherical object with a negative mass $-M$.

a) Write down the line element ds^2 outside this object when there is no matter or energy outside it.

$$ds^2 = -\left(1+\frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1+\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

b) Is there an infinite redshif surface for this object? Explain why.

No, because $-g_{tt}$ is always positive and does not become zero.

c) Is there an event horizon for this object? Explain why.

No, because g_{rr} is always positive and does not become infinite.

d) What is the value of r for which there is a singularity?

For $r = -\frac{2GM}{c^2}$ there is a singularity, namely $g_{rr} \to \frac{1}{0} = \infty$ for this r value.

e) What kind of singularity is this?

Since the singular value of r is not positive, it is a naked singularity. There is also a singularity at $r=0$, which is a coordinate singularity.