GENERAL RELATIVITY

EXAM 2

May 28, 2021

Name:

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1. The metric for a 2-dimensional space is given by

$$ds^2 = \frac{dr^2}{1-r^2/R^2} + r^2 d\phi^2 \; ; \quad R = \text{const.}$$

a) Is this space curved or flat? Why?

) Embed a surface with this metric in a three dimensional Euclidean space described by cylindrical coordinates $r, \phi$, and $z$. (Find $z(r)$ in integral form.)

2. a) Write down the Schwarzschild solution outside a spherical body of mass M.

b) Write down the Einstein field equation to which this $ds^2$ is a solution.

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3. Consider a spherically charged body of mass M and charge Q. When an electron is at a radial position r from the center of this body, the spacetime metric outside it is given by

$$ds^2 = -\left(1 - \frac{GM}{c^2 r} - \frac{e}{m}\frac{k_e Q}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{GM}{c^2 r} - \frac{e}{m}\frac{k_e Q}{c^2 r}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$ , where $k_e$ is the Coulomb constant and m is the electron's mass.

a) Find the value of Q for which the spacetime becomes flat.

b) What is the condition on the radius $R$ of this charged body to become a black hole.

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4. a) Write down the expressions for the absolute gradient (covariant derivative) of a four vector $A^\mu$ and a covector $A_\mu$.

b) Calculate $\nabla_\alpha (A_\mu B^\mu)$

c) Explain why $\nabla_\alpha g^{\mu\nu} = 0$.

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5. Two black holes have masses $M_1 = 3M_\odot$ and $M_2 = 9M_\odot$, where $M_\odot$ is the solar mass.

a) Find the ratio $T_2/T_1$ of the temperatures of the black holes.

b) A black hole which has a temperature of $T_{BH} = 3.7\,K$ at $t=0$ is in contact with the cosmic background radiation. Describe what happens to the temperature of the black hole at a later time.

6. a) Write down the Einstein field equation with the cosmological constant $\Lambda$.

b) Find the expression for $R$ in terms of $\Lambda$ and the trace $T = T^{\mu}{}_{\mu}$ of the energy-momentum tensor $T^{\mu\nu}$.

c) Then express the Einstein equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show your work.

7. The energy-momentum tensor of a perfect fluid is given by
$$T^{\alpha\beta} = \left(\rho + \frac{P}{c^2}\right) U^\alpha U^\beta + P g^{\alpha\beta}$$

a) What is the expression for $T^{\alpha\beta}$ in a locally inertial frame?

b) Write down $T^{\alpha\beta}$ in matrix form in a locally inertial frame.

c) What is $\nabla_\alpha T^{\alpha\beta} = ?$

d) What are the dimensions of $T^{\alpha\beta}$ ?

8. The metric for a 2-sphere of radius $R_o$ is given by $ds^2 = R_o^2 d\theta^2 + R_o^2 \sin^2\theta d\phi^2$.

The $\theta\phi\theta\phi$-component of the Riemann tensor is found to be $R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = R_o^2 \sin^2\theta$

a) Calculate $R_{\theta\theta}$, $R_{\theta\phi}$, $R_{\phi\theta}$, and $R_{\phi\phi}$ components of the Ricci tensor $R_{\beta\nu}$ given by $R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu} = g^{\mu\alpha} R_{\mu\beta\alpha\nu}$

b) Calculate the curvature scalar $R \equiv g^{\alpha\beta} R_{\alpha\beta}$.

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9. a) Write down the squared infinitesimal interval $ds^2$ (the metric) describing the spacetime outside a static spherical distribution of mass $M$.

b) What is the infinitesimal radial distance along a radial line?

c) Calculate the radial distance between two events $A$ and $B$ with coordinates $r_A$ and $r_B$ along such a line in the limit $\frac{2GM}{c^2 r} \ll 1$.

10. In the Yilmaz theory of general relativity, the spacetime geometry outside a spherical object of mass M is described by

$$ds^2 = -e^{-r_s/r}c^2dt^2 + e^{r_s/r}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

where $r_s \equiv 2GM/c^2$.

a) What is the condition that this metric approximates the Schwarzschild metric?

b) Are there black holes according to this metric? Explain your answer.

c) Is there an event horizon? Explain why.

d) If an object falling inward toward $r=0$ crosses $r=r_s$, can this object return back? Explain.