

Ch. 5

Model Universes

In a spatially homogeneous and isotropic universe, the relation between the scale factor $a(t)$, the energy density $\varepsilon(t)$, and the pressure $P(t)$ is given by

the Friedman equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{kc^2}{R_0^2 a^2}$$

the fluid equation

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

and the equation of state

$$P = w \varepsilon$$

The acceleration Eqn:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3P)$$

Evolution of Energy Density:

The universe contains

nonrelativistic matter with $w=0$,
 radiation (relativistic matter) with $w = \frac{1}{3}$,
 dark energy with $w < -\frac{1}{3}$,
 maybe other exotic matter.

* Energy density and pressure for the different components of the universe are additive.

$$\varepsilon = \sum_i \varepsilon_i; \quad P = \sum_i P_i = \sum_i w_i \varepsilon_i$$

The fluid equation

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

$$\sum_i \dot{\epsilon}_i + 3 \frac{\dot{a}}{a} \sum_i (\epsilon_i + p_i) = 0$$

$$\sum_i \dot{\epsilon}_i + 3 \frac{\dot{a}}{a} \sum_i (1 + w_i) \epsilon_i = 0$$

Assuming the different components do not interact with each other then each component i satisfies the equation

$$\dot{\epsilon}_i + 3 \frac{\dot{a}}{a} (1 + w_i) \epsilon_i = 0$$

$$\frac{1}{\epsilon_i} \frac{d\epsilon_i}{dt} + 3 \frac{1}{a} \frac{da}{dt} (1 + w_i) = 0$$

$$\frac{d\epsilon_i}{\epsilon_i} = -3(1 + w_i) \frac{da}{a}$$

Assuming that w_i is constant

$$\ln \epsilon_i = -3(1 + w_i) \ln a + \text{const.}$$

$$\ln \epsilon_i = \ln a^{-3(1 + w_i)} + \text{const.}$$

$$\Rightarrow \epsilon_i = \exp[\text{const} + \ln a^{-3(1 + w_i)}]$$

$$\epsilon_i = \epsilon_{i,0} \cdot a^{-3(1 + w_i)}$$

For nonrelativistic matter, $w_m = 0$

$$\boxed{\epsilon_m = \epsilon_{m,0} a^{-3}}$$

For relativistic matter (radiation) $w_r = \frac{1}{3}$

$$\boxed{\epsilon_r = \epsilon_{r,0} a^{-4}}$$

Understanding a^{-3} & a^{-4} behaviour

Let us write $\mathcal{E} = nE = \frac{N}{V} E$, where N is the number of particles in a volume V , and E is the average energy per particle (nonrelativistic, or relativistic)

$$V = \frac{4\pi}{3} a^3$$

For nonrelativistic particles $E = mc^2 \Rightarrow$

$$\mathcal{E}_m = \frac{N}{\frac{4\pi}{3} a^3} (mc^2) \propto a^{-3}$$

For relativistic particles $E = \frac{hc}{\lambda} \propto \frac{hc}{a}$

$$\Rightarrow \mathcal{E}_r = \frac{N}{\frac{4\pi}{3} a^3} \frac{hc}{\lambda} \propto \frac{1}{a^3} \frac{1}{a} = a^{-4}$$

For photons, it has been assumed that they are not created or destroyed as the universe expands. It can be shown that most of the contribution to $\mathcal{E}_r$ comes from the cosmic microwave background (CMB) rather than other non-CMB radiation from stars.

- * The CMB is a relic of the era when the universe was hot and dense enough to be opaque to photons.
- * If we extrapolate further back, we reach an era when the universe was hot and dense enough to be opaque to neutrinos.
 $\Rightarrow$ There should be cosmic neutrino background (C ν B) today.

Calculations show that the ^{total} neutrino energy density today should be

$$\begin{aligned}\epsilon_\nu &= 3 \times \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \epsilon_{\text{CMB}} \\ &= 0.681 \epsilon_{\text{CMB}},\end{aligned}$$

where we have assumed that there are 3 neutrino types and each ^{one} is relativistic.

Empty Universes:

No matter, radiation, cosmological constant: $\epsilon = 0$

$$\Rightarrow \frac{\ddot{a}}{a^2} = - \frac{k c^2}{R_0^2 a^2} \Rightarrow \ddot{a} = - \frac{k c^2}{R_0^2}$$

i) $k=0$: $\ddot{a}=0 \Rightarrow a(t) = \text{constant}$, which is a static and empty universe.

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

ii) $k=-1$: $\ddot{a} = + \frac{c^2}{R_0^2}$ ($k=+1$ is not possible)

$\ddot{a} = \pm \frac{c}{R_0}$; expansion requires the + sign.

$$\int_0^a da = + \frac{c}{R_0} \int_0^t dt$$

$$a(t) = \frac{c}{R_0} t = \frac{t}{t_0}, \text{ where } t_0 = \frac{R_0}{c}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{1/t_0}{t/t_0} = \frac{1}{t}$$

$$\Rightarrow t_0 = H_0^{-1}$$

A negatively curved, empty and expanding universe is called a Milne universe.

A universe with $k=-1$ and $-\Omega \ll 1$:

$$\frac{\ddot{a}^2}{a^2} = \frac{8\pi G}{c^2} \epsilon - \frac{kc^2}{R_0^2 a^2} ; \quad -\Omega(t) = \frac{\epsilon(t)}{\epsilon_c(t)}$$

$$= \frac{8\pi G}{c^2} \cdot \epsilon_c \cdot -\Omega + \frac{c^2}{R_0^2 a^2} ; \quad (k=-1)$$

$$\dot{a} = \left(\frac{8\pi G}{c^2} \epsilon_c \cdot -\Omega a^2 + \frac{c^2}{R_0^2} \right)^{1/2} \quad \boxed{-\Omega(t) \equiv \frac{\epsilon(t)}{\epsilon_c(t)}}$$

$$= \frac{c}{R_0} (1 + \delta)^{1/2} \approx \frac{c}{R_0} (1 + \frac{1}{2}\delta)$$

$$\int_0^a da \approx \frac{c}{R_0} (1 + \frac{1}{2}\delta) \int_0^t dt = \frac{c}{R_0} (1 + \frac{1}{2}\delta) t$$

$$a(t) \approx (1 + \frac{1}{2}\delta) \frac{t}{t_0} \quad , \text{ where } \delta \equiv \frac{8\pi G \epsilon_c \cdot -\Omega a^2}{c^2 \times \frac{R_0^2}{c^2}} \ll 1, \quad -\Omega \ll 1$$

$$\approx \frac{t}{t_0}$$

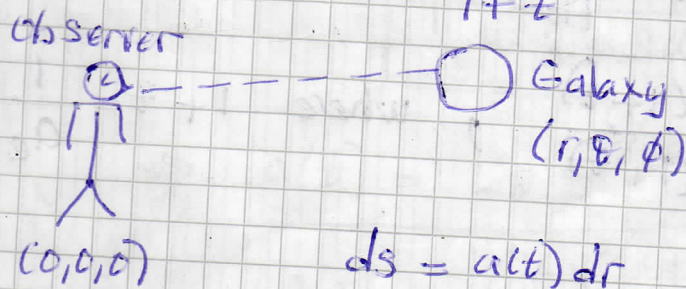
In an empty expanding universe

$$1+z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = \frac{1}{t_e/t_0} = \frac{t_0}{t_e}$$

The observed light from the source was emitted when

$$t_e = \frac{t_0}{1+z} = \frac{H_0^{-1}}{1+z}$$

See Hubble
p. 382



$$ds = a(t) dr \quad d\theta = d\phi = 0$$

$$\text{Proper distance } dp(t) = \int a(t) dr$$

$$dp(t_0) = a(t_0) \int_0^r dr = r$$

If light is emitted at time t_e and observed at time t_o , the null geodesic followed by the light satisfies

$$ds^2 = 0 = -c^2 dt^2 + a^2 dr^2 \quad (d\theta = d\phi = 0)$$

$$\Rightarrow c \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^r dr = r$$

$$\text{since } d_p(t_o) = r \Rightarrow d_p(t_o) = c \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

in an empty expanding universe $a(t) = \frac{t}{t_o}$

$$\Rightarrow d_p(t_o) = c t_o \int_{t_e}^{t_o} \frac{dt}{t} = c t_o \ln\left(\frac{t_o}{t_e}\right)$$

$$\text{since } 1+z = \frac{t_o}{t_e} \Rightarrow d_p(t_o) = \frac{c}{H_0} \ln(1+z)$$

$$\text{From } d_p(t) = \int a(t) dr$$

$$d_p(t_e) = a(t_e) \int_0^r dr = a(t_e) r ; r = d_p(t_o)$$

$$= a(t_e) \cdot d_p(t_o)$$

$$= \left(\frac{1}{1+z}\right) \cdot \frac{c}{H_0} \ln(1+z)$$

$$d_p(t_e) = \frac{c}{H_0} \frac{\ln(1+z)}{(1+z)} , \text{ where } 1+z = \frac{1}{a_e} \text{ was used.}$$

Single-component Universes:

Consider a $k=0$ (flat) universe with a single component of w . The Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2} \epsilon, \text{ where}$$

$$\epsilon(a) = \epsilon_0 a^{-3(1+w)}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \epsilon_0 a^{-3-3w+2}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \epsilon_0 a^{-(1+3w)}$$

Guess: $a(t) = C_0 t^q$

$$\dot{a} = C_0 q t^{q-1}; \quad \dot{a}^2 = C_0^2 q^2 t^{2q-2}$$

$$C_0^2 q^2 t^{2q-2} = \frac{8\pi G}{3c^2} \epsilon_0 t^{-q(1+3w)} \cdot C_0^{-(1+3w)}$$

$$C_0^2 q^2 t^{(2q-2+q+3wq)} = \frac{8\pi G}{3c^2} \epsilon_0 C_0^{-(1+3w)}$$

$$C_0^2 q^2 t^{q(3+3w)-2} = \frac{8\pi G}{3c^2} \epsilon_0 C_0^{-(1+3w)}$$

$$\Rightarrow q(3+3w)-2 = 0$$

$$q = \frac{2}{3+3w}$$

$$a(t_0) = 1 = C_0 t_0^{2/(3+3w)} \\ \Rightarrow C_0 = t_0^{-2/(3+3w)}$$

$$\Rightarrow a(t) = C_0 t^{2/(3+3w)}$$

$$a(t) = \left(\frac{t}{t_0} \right)^{2/(3+3w)}; \quad w \neq -1$$

$$\text{Calculating } C_0: C_0^2 \left(\frac{2}{3+3w} \right)^2 = \frac{8\pi G}{3c^2} \epsilon_0 C_0^{-(1+3w)}$$

$$C_0^{(3+3w)} \cdot \frac{4}{9} \frac{1}{(1+w)^2} = \frac{8\pi G}{3c^2} \epsilon_0$$

$$\text{using } C_0 = \left(\frac{1}{t_0} \right)^{2/(3+3w)}$$

← from (4), putting $a(t_0) = 1 = C_0 t_0^{2/c}$

$$\left(\frac{1}{t_0} \right)^2 \frac{1}{(1+w)^2} = \frac{6\pi G}{c^2} \epsilon_0$$

$$\Rightarrow t_0 = \frac{1}{(1+w)} \left(\frac{c^2}{6\pi G \epsilon_0} \right)^{1/2}$$

$$\text{since } H_0^2 = \frac{8\pi G}{3c^2} \epsilon_0 = \frac{1}{3} \frac{8}{6} \left(\frac{6\pi G \epsilon_0}{c^2} \right) = \frac{4}{9} \left(\right)$$

$$H_0 = \frac{2}{3} \left(\right)^{1/2}$$

$$H_0^{-1} = \left(\frac{2}{3} \right)^{-1} \left(\frac{c^2}{6\pi G \epsilon_0} \right)^{1/2}$$

$$\Rightarrow t_0 = \frac{2}{3(1+w)} H_0^{-1} \quad ; \quad w \neq -1$$

(XX)

$$* \quad t_0 > H_0^{-1} \text{ (Hubble time) if } \frac{2}{3(1+w)} > 1$$

$$3 + 3w < 2 \Rightarrow w < -\frac{1}{3}$$

The Universe is older than the Hubble time if $w < -\frac{1}{3}$

$$* \quad t_0 < H_0^{-1} \text{ if } \frac{2}{3(1+w)} < 1$$

$$3 + 3w > 2 \Rightarrow w > -\frac{1}{3}$$

(for example for $w=0$ and $w=1/3$)

Since $\epsilon(a) = \epsilon_0 a^{-3(1+w)}$; $a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$

$$\boxed{\epsilon(t) = \epsilon_0 \left(\frac{t}{t_0}\right)^{-2}}$$

The energy density or ρ decreases as $\frac{1}{t^2}$.

Since in a $k=0$ (flat) universe with $\Lambda=0$

$$\epsilon_0 = \epsilon_{e,p} = \frac{3c^2}{8\pi G} H_0^2 ; \text{ using } (X) \Rightarrow$$

$$= \frac{3c^2}{8\pi G} \cdot \frac{4}{9(1+w)^2} t_0^{-2}$$

$$= \frac{c^2}{6\pi G (1+w)^2} t_0^{-2}$$

$$\Rightarrow \epsilon(t) = \frac{1}{6\pi (1+w)^2} \cdot \frac{c^2}{G} t^{-2}$$

Note : $\frac{E_p}{l_p^3} \frac{1}{t_p^{-2}} = \left(\frac{\hbar c}{G}\right)^{1/2} c^2 \frac{1}{\left(\frac{G\hbar}{c^3}\right)^{3/2}} \frac{1}{\left(\frac{G\hbar}{c^5}\right)^{-1}}$

P : Planck

$$= \frac{c^2}{G}$$

$$\Rightarrow \epsilon(t) = \frac{1}{6\pi (1+w)^2} \frac{E_p}{l_p^3} \left(\frac{t}{t_p}\right)^{-2}$$

$$\boxed{\epsilon(t) = \frac{1}{6\pi (1+w)^2} \frac{E_p t_p^2}{l_p^3} t^{-2}}$$

Using $1+z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t_0)} = \left(\frac{t_0}{t}\right)^{2/(3+3w)}$

↑ from 2 pages before

The time t_e at which light from a distant galaxy was emitted is given by

$$t_e = \frac{t_0}{(1+z)^{3(1+w)/2}}, \quad (\text{***})$$

using $t_0 = \frac{2}{3(1+w)} H_0^{-1}$

$$t_e = \frac{2}{3(1+w)(1+z)^{3(1+w)/2}} \cdot H_0$$

The current proper distance to the galaxy is

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c \int_{t_e}^{t_0} \left(\frac{t}{t_0} \right)^{-2/(3+3w)} dt$$

$$x = \frac{t}{t_0}, \quad dt = t_0 dx$$

$$d_p(t_0) = c t_0 \int_{\frac{t_e}{t_0}}^1 x^{-2/(3+3w)} dx$$

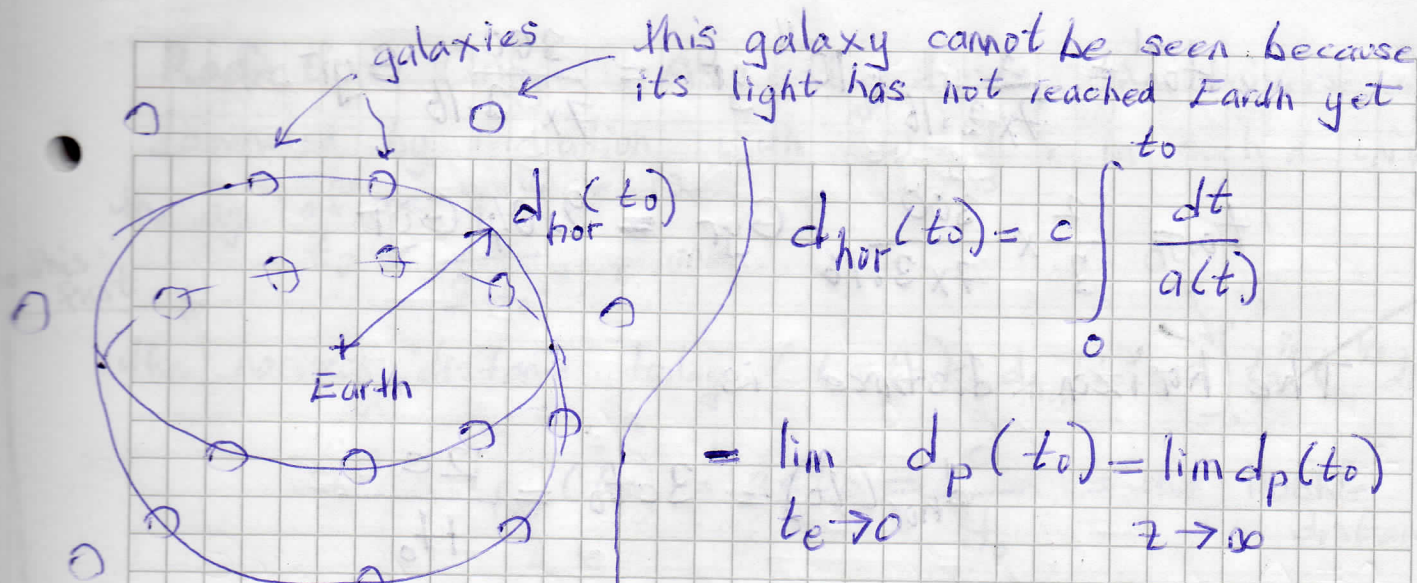
$$d_p(t_0) = c t_0 \frac{3(1+w)}{(1+3w)} \left[1 - \left(\frac{t_e}{t_0} \right)^{(1+3w)/(3+3w)} \right]$$

$$, w \neq -\frac{1}{3}$$

using (***) and (**) $\Rightarrow$

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{(1+3w)} \left[1 - (1+z)^{-(1+3w)/2} \right]$$

Definition: The particle horizon distance is the proper distance (at the time of observation) to an object that emitted a light at $t=0$.



$$d_{\text{hor}}(t_0) = c \int_0^{t_0} \frac{dt}{a(t)}$$

$$= \lim_{t_e \rightarrow 0} d_p(t_0) = \lim_{z \rightarrow \infty} d_p(t_0)$$

the horizon (a spherical surface with radius $d_{\text{hor}}(t_0)$)

$$d_{\text{hor}}(t_0) = c t_0 \frac{3(1+w)}{1+3w} = \frac{c}{H_0} \frac{2}{(1+3w)}$$

$$w \neq -\frac{1}{3}$$

$$d_{\text{hor}}(t) = a(t) c \int_0^t \frac{dt}{a(t)}$$

$$d_{\text{hor}}(t_0) = a(t_0) c \int_0^{t_0} \frac{dt}{a(t)} = c \int_0^{t_0} \frac{dt}{a(t)}$$

* in a flat universe dominated by matter ($w=0$) or by radiation ($w=\frac{1}{3}$) an observer

can see only a finite portion of the infinite volume of the universe.

The visible universe is the three dimensional space lying within the horizon. It consists of all points that are causally connected to the observer by light signals.

* In a universe with $w \leq -\frac{1}{3}$ $d_{\text{hor}}(t_0) = \infty$

Matter Only: Consider a flat universe containing nonrelativistic matter only with $w=0$.

Age of the universe $t_0 = \frac{2}{3H_0}$

$$H_0 = 70 \text{ s}^{-1} \text{ km Mpc}^{-1} = 70 \text{ s}^{-1} \text{ km} (3 \times 10^{19} \text{ km})^{-1}$$

$$= 2.3 \times 10^{-18} \text{ s}^{-1}; H_0^{-1} = \frac{3}{7} \times 10^{18} \text{ s} = \frac{3}{7} \times 10^{18} \frac{\text{yr}}{3.16 \times 10^7}$$

$$H_0^{-1} = \frac{3}{7 \times 3.16} \times 10^{11} \text{ yr} = \frac{300}{7 \times 3.16} \text{ Gyr}$$

$$t_0 = \frac{2}{3} \times \frac{300}{7 \times 3.16} \text{ Gyr} = 9.04 \text{ Gyr}$$

The horizon distance is

$$d_{\text{hor}}(t_0) = 3ct_0 = \frac{2c}{H_0}$$

The scale factor is

$$a(t)_{\text{mat}} = \left(\frac{t}{t_0}\right)^{2/3}$$

The proper distance to a galaxy with redshift z ,

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{2/3}} = 3ct_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{1/3} \right]$$

$$= \frac{2c}{H_0} \left[1 - \frac{1}{1+z} \right]$$

$$\text{From } ds = a dr \Rightarrow \frac{d_p(t_0)}{d_p(t_e)} = \frac{\int a(t_0) dr}{\int a(t_e) dr}$$

$$= \frac{a(t_0) \int dr}{a(t_e) \int dr} = 1+z$$

$$\Rightarrow d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{2c}{H_0(1+z)} \left[1 - \frac{1}{1+z} \right]$$

* A matter-dominated flat universe is called an Einstein-de Sitter universe.

Radiation Only: At early times, the universe was dominated by radiation with $w = \frac{1}{3}$. In such a universe the age of the universe is $t_0 = \frac{1}{2H_0}$, and $a(t_0) = 1$

the horizon distance today ($t = t_0$); $d_{hor}(t_0) = c \int_0^{t_0} t^{-1/2} dt$

$$d_{hor}(t_0) = 2ct_0 = \frac{c}{H_0} = \text{the Hubble distance}$$

The scale factor is

$$a(t)_{rad} = \left(\frac{t}{t_0}\right)^{2/(3+3 \cdot \frac{1}{3})} = \left(\frac{t}{t_0}\right)^{1/2}$$

The proper distance to a light source (galaxy) at time t_0 is

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{1/2}}$$

$$= 2ct_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{1/2} \right] = \frac{c}{H_0} \frac{t}{1+z}$$

The proper distance at the time the light was emitted

was

$$\frac{d_p(t_e)}{d_p(t_0)} = \frac{1}{1+z} \Rightarrow d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{c}{H_0} \frac{t}{(1+z)^2}$$

The energy density in a flat, radiation dominated universe is

$$\epsilon_r(t) = \frac{1}{6\pi(1+\frac{1}{3})^2} \frac{E_p}{L_p^3} \left(\frac{t}{t_p}\right)^{-2} = \frac{3}{32\pi} \frac{E_p}{L_p^3} \left(\frac{t}{t_p}\right)^{-2}$$

Using $\epsilon_r = \alpha T^4 = \left(\frac{\pi^2 k_B^4}{15 \hbar^3 c^3}\right) T^4$

$$\Rightarrow T^4 = \frac{15 \hbar^3 c^3}{\pi^2 k_B^4} \cdot \frac{3}{32\pi} \left(\frac{k_B}{E}\right)^{1/2} c^2 \frac{1}{\left(\frac{E \hbar}{c^3}\right)^{3/2}} \left(\frac{t}{t_p}\right)^{-2}$$

$$\Rightarrow T(t) = \left(\frac{45}{32\pi^3} \right)^{1/4} \frac{1}{k_B} \left(\frac{\hbar c^5}{G} \right)^{1/2} \left(\frac{t}{t_p} \right)^{-1/2}$$

$$\underbrace{\frac{1}{k_B} \left(\frac{\hbar c^5}{G} \right)^{1/2}}_{\frac{E_p}{k_B} \equiv T_p}$$

$$T(t) = \left(\frac{45}{32\pi^3} \right)^{1/4} T_p \left(\frac{t}{t_p} \right)^{-1/2}$$

Recall, the number density of photons in blackbody radiation is

$$n_\gamma = \left(\frac{2.4041}{\pi^2 \hbar^3 c^3} k_B^3 \right) T(t)^3,$$

then the mean energy per photon is

$$E_{\text{mean}}(t) = \frac{E_r(t)}{n_\gamma} = \frac{2.404}{\pi^2 \hbar^3 c^3}$$

$$= \left(\frac{\pi^2}{15 \hbar^3 c^3} \cdot \frac{\pi^2 \hbar^3 c^3}{2.404} \right) k_B T(t)$$

$$= \frac{\pi^4}{15(2.404)} k_B T = 2.6958 k_B T$$

$$\approx 2.7 k_B T(t)$$

$$\approx 2.7 \left(\frac{45}{32\pi^3} \right)^{1/4} k_B T_p \left(\frac{t}{t_p} \right)^{-1/2}$$

$$\approx \left(\quad \right) E_p \left(\frac{t}{t_p} \right)^{-1/2}$$

Note that in a flat radiation dominated universe, as $t \rightarrow 0$

$$T(t) \rightarrow \infty, \quad E_r(t) \rightarrow \infty, \quad n_\gamma(t) \rightarrow \infty, \quad E_{\text{mean}}(t) \rightarrow \infty.$$

Should we take these infinities seriously?

No, general relativity is believed to be valid for times after the Planck time $t_p = 5.39 \times 10^{-44} \text{ s}$ and energies less than the Planck energy $E_p = 1.96 \times 10^9 \text{ J} = 1.22 \times 10^{19} \text{ GeV}$

General relativity is a classical theory, which does not take into account the effects of quantum mechanics.

In a flat, radiation dominated universe, the horizon distance

$$d_{\text{hor}}(t) = \int a(t) dr = a(t) \int dr$$

$$\rightarrow d_{\text{hor}}(t) = a(t) \int_0^t \frac{dt}{a(t)} = a(t) \int_0^t \frac{t_0^{1/2}}{t^{1/2}} dt = 2ct = 2l_p \left(\frac{t}{t_p} \right)$$

$\triangleright a(t) = \left(\frac{t}{t_0} \right)^{1/2}$

the volume of the visible universe at time t is

$$V_{\text{hor}}(t) = \frac{4\pi}{3} d_{\text{hor}}^3(t) = \frac{4\pi}{3} (2l_p)^3 \left(\frac{t}{t_p} \right)^3$$

The number of photons inside the horizon at time t is

$$N(t) = V_{\text{hor}}(t) n(t)$$

$$= \frac{32\pi}{3} c^3 t_0^{3/2} \left(\frac{2.4}{\pi^2} \frac{k_B^3}{h^3 c^3} \right) T(t)^3 t^{3/2}$$

$$= \frac{32(2.4)^3}{3\pi} \frac{k_B^3}{h^3} \left(\frac{45}{32\pi^3} \right)^{3/4} T_p^3 \left(\frac{t}{t_p} \right)^{-6} t^{3/2} t_0^{3/2}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 dr^2 = 0$$

$$c dt / a(t) = dr$$

(The de Sitter Universe)

Lambda Only: Consider a flat universe in which the energy density is contributed by a cosmological constant Λ . Then

$$\frac{\ddot{a}}{a^2} = \frac{8\pi G}{3c^2} \epsilon_{\Lambda}$$

$$= \frac{\Lambda}{3} \quad ; \quad \Lambda = \frac{8\pi G}{c^2} \epsilon_{\Lambda}$$

$$\Rightarrow \ddot{a}(t) = \left(\frac{8\pi G}{3c^2} \epsilon_{\Lambda} \right) a^2$$

$$= H_0^2 a^2, \text{ where } H_0 = \left(\frac{8\pi G}{3c^2} \epsilon_{\Lambda} \right)^{1/2}$$

$$\dot{a}(t) = H_0 a(t)$$

$$\int_0^a \frac{da}{a} = \int H_0 dt$$

$$\ln a = H_0 t + C_0$$

$$a(t) = e^{C_0} e^{H_0 t}$$

$$a(t_0) = e^{H_0 t_0 + C_0} = 1 \Rightarrow C_0 = -H_0 t_0$$

$$\Rightarrow \boxed{a(t) = e^{H_0(t-t_0)}}$$

Remark: The steady state model:

$$v(t) = \frac{dr(t)}{dt} = H r(t) \quad ; \quad r(t) = a(t) r$$

comoving

$$\frac{d}{dt}(a(t) \cdot r) = H a(t) r \quad ; \quad H = \text{constant} = H_0$$

$$\frac{da}{a} = H_0 dt \Rightarrow \ln a = H_0 t + \text{const.}$$

$$\Rightarrow a(t) = e^{H_0(t-t_0)}$$

$a(t) \rightarrow 0$ as $t \rightarrow -\infty$. The universe is infinitely old. #

$$\begin{aligned} d_{\text{Hor}}(t) &= c \int_{-\infty}^t \frac{dt}{a(t)} = e^{+H_0 t_0} \int_{-\infty}^t e^{-H_0 t} dt \\ &= c e^{+H_0 t_0} \left(\frac{1}{(-H_0)} (e^{-H_0 t} - \infty) \right) = \\ &= \infty \end{aligned}$$

The proper distance to a light source is

$$\begin{aligned} d_p(t_0) &= c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c e^{H_0 t_0} \int_{t_e}^{t_0} e^{-H_0 t} dt \\ &= c e^{H_0 t_0} \left(-\frac{1}{H_0} \right) (e^{-H_0 t_0} - e^{-H_0 t_e}) \\ &= \frac{c}{H_0} [e^{H_0(t_0 - t_e)} - 1] \end{aligned}$$

since $1+z = \frac{1}{a(t_e)} = e^{H_0(t_e - t_0)} \Rightarrow$

$$d_p(t_0) = \frac{c}{H_0} z,$$

and $d_p(t_e) = \frac{c}{H_0} \left(\frac{z}{1+z} \right)$

Multiple - component Universes:

The Friedmann equation

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \epsilon(t) - \frac{kc^2}{R_0^2 a(t)^2}$$

where $\epsilon(t) = \epsilon_r(t) + \epsilon_m(t) + \epsilon_\Lambda(t) + \dots$

the critical density $\epsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2$

the density parameter $\Omega(t) = \frac{\epsilon(t)}{\epsilon_c(t)}$

$$= \frac{\epsilon_r}{\epsilon_c} + \frac{\epsilon_m}{\epsilon_c} + \frac{\epsilon_\Lambda}{\epsilon_c} + \dots$$

$$\Omega(t) = \Omega_r(t) + \Omega_m(t) + \Omega_\Lambda(t) + \dots$$

The Friedmann equation

$$1 = \frac{8\pi G}{3c^2 H^2(t)} \cdot \epsilon(t) - \frac{kc^2}{R_0^2 a^2 H^2}$$

$$= \frac{\epsilon(t)}{\epsilon_c(t)} - \frac{kc^2}{R_0^2 a(t)^2 H(t)^2}$$

$$1 - \Omega(t) = - \frac{kc^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\text{or } \frac{k}{R_0^2} = \left(-\Omega(t_0) - 1 \right) \frac{H(t_0)^2}{c^2}$$

$\frac{k}{R_0^2} = \frac{H_0^2}{c^2} (-\Omega_0 - 1)$, using this the Friedmann equation can be written as

$$H(t)^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \epsilon(t) - \frac{H_0^2}{a(t)^2} (-\Omega_0 - 1)$$

Dividing by H_0^2 , this becomes

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{c,0}} + \frac{1 - \Omega_0}{a(t)^2},$$

where $\epsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G}$

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon_r(t)}{\epsilon_{c,0}} + \frac{\epsilon_m(t)}{\epsilon_{c,0}} + \frac{\epsilon_\Lambda(t)}{\epsilon_{c,0}} + \frac{1 - \Omega_0}{a^2}$$

From $\epsilon(a) = \epsilon_0 \frac{1}{a^{3(1+w)}}$

$$\epsilon_m(t) = \epsilon_{m,0} \frac{1}{a^3}, \quad w=0$$

$$\epsilon_r(t) = \epsilon_{r,0} \frac{1}{a^4}, \quad w = \frac{1}{3}$$

$$\epsilon_\Lambda(t) = \epsilon_{\Lambda,0}$$

$$\Rightarrow \frac{H(t)^2}{H_0^2} = \frac{\epsilon_{r,0}}{\epsilon_{c,0}} \frac{1}{a^4} + \frac{\epsilon_{m,0}}{\epsilon_{c,0}} \frac{1}{a^3} + \frac{\epsilon_{\Lambda,0}}{\epsilon_{c,0}} + \frac{1 - \Omega_0}{a^2}$$

$$\boxed{\frac{H(t)^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2}}$$

and $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$

, it has been assumed that the components of the universe do not interact with each other.

Using $H(t) = \frac{\dot{a}(t)}{a(t)}$ and taking the square root.

$$H_0^{-1} \dot{a}(t) = \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0) \right]^{1/2}$$

$$\frac{da}{dt} \frac{1}{H_0} = \left[\right]^{1/2}$$

$$\int_0^a \frac{da}{\left[\right]^{1/2}} = H_0 \int_0^t dt = H_0 t$$

$$H_0 t = \int_0^a \frac{da}{\left[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0) \right]^{1/2}}$$

This integral can be done numerically.

Remark: $\Omega_{m,0} = \underbrace{\Omega_{b,0}}_{\substack{\text{baryonic matter} \\ \approx 0.05}} + \underbrace{\Omega_{dm,0}}_{\substack{\text{dark matter} \\ \approx 0.27}} \approx 0.31 - 0.32$

Ryden assumes: $\Omega_{\Lambda,0} = \underbrace{\Omega_{de,0}}_{\substack{\text{dark energy} \\ \approx 0.68 - 0.69}}$

~~Ryden assumes implicitly~~

$$\Omega_{r,0} = 9.0 \times 10^{-5}$$

$$\Omega_0 = 1$$

Ryden calls this the Benchmark Model

Matter + Curvature

Consider a universe containing only nonrelativistic pressureless matter, with $w=0$. If such a universe is spatially flat, then

$$a(t) = \left(\frac{t}{t_0} \right)^{2/3} ; k=0$$

→ universe expands for ever

Temperature of universe $T(t)$ decreases with time
such a fate is called the "Big Chill".

* if the universe is curved ($k = +1$ or -1)

$$\rho_0 = \frac{\rho}{m_p} = \frac{H(t)^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2} \quad \text{curvature}$$

this universe will cease to expand when $H(t) = 0$

$$\Rightarrow \quad 0 = \frac{R_0}{a_{\max}^3} + \frac{1 - R_0}{a_{\max}^2}$$

$$\Rightarrow a_{\max} = \frac{\rho_0}{-\rho_0^{-1}} \Rightarrow -\rho_0 > 1 \text{ so that } a_{\max} \text{ is positive.}$$

$$\Rightarrow k = +1$$

After reaching $a_{\max}$, $a(t)$ will start contracting and eventually the universe will collapse to $a=0$.

This event is called the Big Crunch.

Matter dominated universe with $k = -1 \Rightarrow \Omega_0 < 1$

$$\frac{1}{H_0^2} \ddot{a}^2(t) = \frac{\Omega_0}{a(t)} + \frac{1 - \Omega_0}{1} > 0$$

$\Rightarrow \dot{a}(t) > 0 \Rightarrow$ the universe will expand forever.

In a curved universe containing only matter, the Friedmann equation can be written as

$$\frac{\dot{a}^2}{H_0^2} = \frac{\Omega_0}{a} + (1 - \Omega_0)$$

$$\frac{1}{H_0^2} \frac{da}{dt} = \sqrt{\frac{\Omega_0}{a} + (1 - \Omega_0)}$$

$$H_0 \int_0^t dt = \int_0^a \frac{da}{\left[\frac{\Omega_0}{a} + 1 - \Omega_0 \right]^{1/2}}$$

$$H_0 t = \int_0^a \frac{da}{\left[\frac{\Omega_0}{a} + 1 - \Omega_0 \right]^{1/2}}$$

$$\Rightarrow i) a(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_0} \right)^{2/3} t^{2/3} = \left(\frac{t}{t_0} \right)^{2/3}$$

$$\text{, where } t_0 = \left(\frac{3}{2} H_0 \sqrt{\Omega_0} \right)^{-1} ; \text{ for } \Omega_0 = 1 \text{ (} k=0 \text{)}$$

$$ii) a(\theta) = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} (1 - \cos \theta)$$

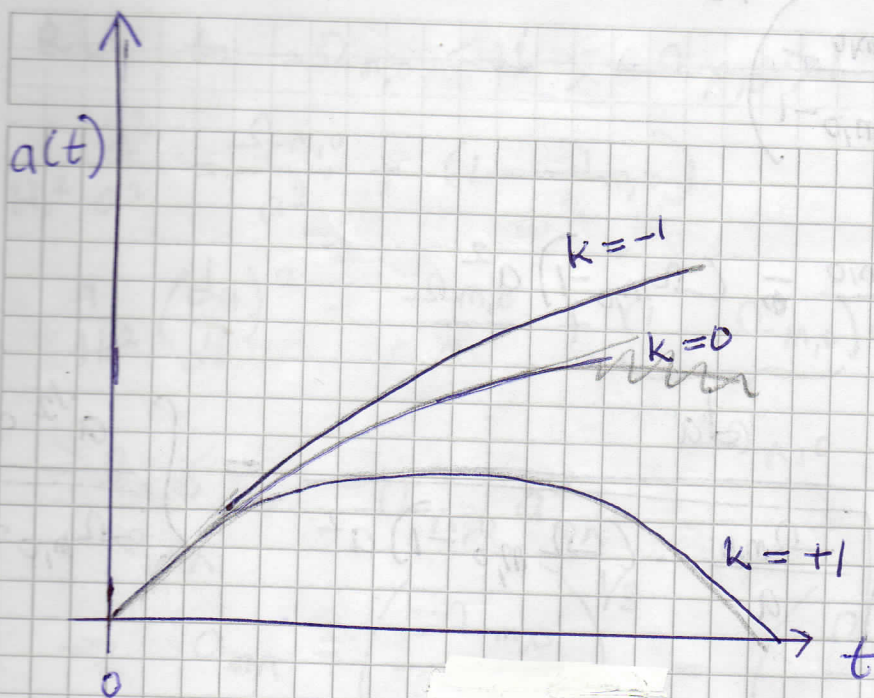
$$t(\theta) = \frac{1}{2 H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\theta - \sin \theta) ; 0 \leq \theta \leq 2\pi$$

$$\left. \begin{aligned} \text{At } \theta = 2\pi, t = t_{\text{crunch}} \\ = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \end{aligned} \right\} \begin{aligned} &\text{for } k=1 \\ &\Omega_0 > 1 \\ &\text{For } \Omega_0 = 1.1 \\ &t_{\text{crunch}} = 109.28 H_0^{-1} \\ &a(t_{\text{crunch}}) = 0 \end{aligned}$$

$$iii) \text{ For } \Omega_0 < 1 ; k = -1$$

$$a(\eta) = \frac{1}{2} \frac{\Omega_0}{1 - \Omega_0} (\cosh \eta - 1)$$

$$t(\eta) = \frac{1}{2 H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} (\sinh \eta - \eta)$$



Matter + Lambda:

Consider a spatially flat universe containing both matter and a cosmological constant. (Such a universe is a close approximation to our universe today.)

$$\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0} = 1$$

$\ll 1$

$$\Rightarrow \Omega_{\Lambda,0} \approx 1 - \Omega_{m,0}$$

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2}$$

$\ll 1$

$$\approx \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

$\propto$ if $\Omega_{m,0} < 1 \Rightarrow \Omega_{\Lambda,0} > 0$ and the universe will continue to expand if it is expanding at $t = t_0$.
(go to the page after the next)

** if $\Omega_{m,0} > 1 \Rightarrow \Omega_{\Lambda,0} < 0$ (negative cosmological constant)

$$\frac{1}{H_0^2} \ddot{a}^2 \approx \frac{\Omega_{m,0}}{a} - (\Omega_{m,0} - 1)a^2 = 0 \Rightarrow \text{next page}$$

$$a_{\max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0}-1} \right)^{1/3}$$

$$\frac{1}{H_0^2} \left(\frac{da}{dt} \right)^2 \approx \frac{\Omega_{m,0}}{a} - (\Omega_{m,0}-1)a^2$$

$$H_0 \int_0^t dt = \int \frac{da}{\sqrt{\frac{\Omega_{m,0}}{a} - (\Omega_{m,0}-1)a^2}} = \int \frac{a^{1/2} da}{\sqrt{\Omega_{m,0} - (\Omega_{m,0}-1)a^3}}$$

$$= \frac{2}{3} \int da^{3/2} \sqrt{\Omega_{m,0} - \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}} \right) (a^{3/2})^2}$$

$$\text{Let } x = \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}} \right)^{1/2} a^{3/2} ; dx = \left(\right)^{1/2} da^{3/2}$$

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0}}} \int_0^x \frac{dx}{\sqrt{1-x^2}}$$

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0}}} \sin^{-1} x = \frac{2}{3\sqrt{\Omega_{m,0}}} \sin^{-1} \left[\left(\frac{a}{a_{\max}} \right)^{3/2} \right]$$

$$\text{At } a=0, \sin^{-1}[0] = 0 \text{ or } \pi$$

$$\Rightarrow t=0 \text{ or } t=t_{\text{crunch}} = \frac{2\pi}{H_0 3\sqrt{\Omega_{m,0}-1}} \left. \begin{array}{l} \text{For } \Omega_{m,0}=1.1 \\ t_{\text{crunch}} = 6.62 H_0^{-1} \\ a=0 \end{array} \right\}$$

Thus a flat universe with $\Omega_{m,0} < 1$ ends in a Big Crunch. (go to Matter + Curvature + Lambda)

Back to $\Omega_{m,0} < 1$, $\Omega_{\Lambda,0} > 0$:

$$\frac{1}{H_0^2} \frac{\ddot{a}^2}{a^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

$$\frac{1}{H_0^2} \left(\frac{da}{dt} \right)^2 = \frac{\Omega_{m,0}}{a} + \underbrace{(1 - \Omega_{m,0})}_{-\Omega_{\Lambda,0}} a^2$$

$$\frac{\Omega_{m,0}}{a_{mn}} = \Omega_{\Lambda,0} a_{mn}^2$$

$$\Rightarrow a_{mn} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} \right)^{1/3} = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}} \right)^{1/3}$$

At $a = a_{mn}$ $\frac{\Omega_{m,0}}{a_{mn}^3} = \Omega_{\Lambda,0}$

$$H_0 \int dt = \int \frac{da}{\sqrt{-\Omega_{m,0}/a + \Omega_{\Lambda,0} a^2}}$$

$$= \frac{1}{\sqrt{-\Omega_{\Lambda,0}}} \int \frac{a da}{\sqrt{a_{mn}^3 + a^3}} = \frac{1}{\sqrt{-\Omega_{\Lambda,0}}} \frac{2}{3} \int \frac{da^{3/2}}{a_{mn}^{3/2} \sqrt{1 + \left(\frac{a}{a_{mn}} \right)^3}}$$

$$= \frac{2}{3\sqrt{-\Omega_{\Lambda,0}}} \int \frac{dx}{\sqrt{1+x^2}}; \quad x = \left(\frac{a}{a_{mn}} \right)^{3/2}$$

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \sinh^{-1} \left[\left(\frac{a}{a_{mn}} \right)^{3/2} \right]$$

At early times, when $a \ll a_{mn}$

$$\Rightarrow \frac{1}{H_0} H_0 t \approx \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \left(\frac{a}{a_{mn}} \right)^{3/2} = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} \cdot \frac{3}{2} a^{3/2}$$

$$\Rightarrow a(t) \approx \left(\frac{3}{2} \sqrt{1 - \Omega_{m,0}} H_0 t \right)^{2/3} \propto t^{2/3}$$

At late times, when $a \gg a_{mn}$

$$\left(\frac{a}{a_{mn}}\right)^{3/2} \approx \sinh\left(\frac{3}{2}\sqrt{-\Omega_{\Lambda 0}} H_0 t\right)$$

$$= \sinh(y) = \frac{e^y - e^{-y}}{2} \approx \frac{e^y}{2} \quad ; y \gg 1$$

$$\left(\frac{a}{a_{mn}}\right)^{3/2} \approx \frac{1}{2} \exp\left(\frac{3}{2}\sqrt{-\Omega_{\Lambda 0}} H_0 t\right)$$

$$a \approx \left[\frac{1}{2} a_{mn}^{3/2} \exp\left(\frac{3}{2}\sqrt{-\Omega_{\Lambda 0}} H_0 t\right) \right]^{2/3}$$

$$a(t) \approx \left(\frac{1}{2}\right)^{2/3} a_{mn} \exp\left(\sqrt{-\Omega_{\Lambda 0}} H_0 t\right)$$

Age of the universe at $a = a_{mn}$

$$H_0 t_{mn} \approx \frac{2}{3\sqrt{1-\Omega_{\Lambda 0}}} \sinh^{-1}\left[\left(\frac{a_{mn}}{a_{mn}}\right)^{3/2}\right] = \frac{2}{3\sqrt{1-\Omega_{\Lambda 0}}} \sinh^{-1}\left[\left(\frac{1-\Omega_{\Lambda 0}}{-\Omega_{\Lambda 0}}\right)^{1/2}\right]$$

$$t_{mn} = \frac{2 H_0^{-1}}{3\sqrt{1-\Omega_{\Lambda 0}}} \underbrace{\sinh^{-1}(1)}_{0.88} = \frac{0.588}{\sqrt{1-0.31}} H_0^{-1}$$

$$t_{mn} \approx 0.707 H_0^{-1} \approx (10.17 \pm 0.30) \text{ Gyr}$$

The current age of the universe is

$$t_0 \approx \frac{2 H_0^{-1}}{3\sqrt{1-\Omega_{\Lambda 0}}} \sinh^{-1}\left[\left(\frac{1-\Omega_{\Lambda 0}}{-\Omega_{\Lambda 0}}\right)^{1/2}\right] \approx (13.74 \pm 0.40) \text{ Gyr}$$

So, in this model the cosmological constant has been the dominant component of the universe since for the last 3.6 billion years.
(go to 4 pages before to $\Omega_{\Lambda 0} > 1$ case)

Matter + Curvature + Lambda:

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{ro}}{a^4} + \frac{\Omega_{mo}}{a^3} + \frac{1 - \Omega_0}{a^2} + \Omega_{\Lambda,0}$$

Assume $\Omega_0 = \Omega_{mo} + \Omega_{\Lambda,0} > 1$

$$\Rightarrow \frac{H(t)^2}{H_0^2} = \frac{\Omega_{mo}}{a^3} + \frac{1 - \Omega_{mo} - \Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0}$$

$$\frac{\dot{a}^2(t)}{H_0^2} = \frac{\Omega_{mo}}{a^2} + 1 - \Omega_{mo} - \Omega_{\Lambda,0} + \Omega_{\Lambda,0} a^2 = 0$$

$$\Rightarrow a^4 - \Omega_{\Lambda,0} - (\Omega_{mo} + \Omega_{\Lambda,0} - 1) a^2 + \Omega_{mo} = 0$$

> 1

Let $a^2 = b \Rightarrow$

$$b^2 - \left(\frac{\Omega_{ro} + \Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right) b + \frac{\Omega_{ro}}{\Omega_{\Lambda,0}} = 0$$

$$a^2 = b = \frac{\frac{\Omega_{ro} + \Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \pm \sqrt{\left(\frac{\Omega_{ro} + \Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right)^2 - 4 \frac{\Omega_{ro}}{\Omega_{\Lambda,0}}}}{2}$$

$$a = b^{1/2} = \left[\frac{1}{2} \left(\frac{\Omega_{ro} + \Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \pm \sqrt{\left(\frac{\Omega_{ro} + \Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right)^2 - 4 \frac{\Omega_{ro}}{\Omega_{\Lambda,0}}} \right) \right]^{1/2}$$

Assume $\Omega_{ro} \ll \Omega_{\Lambda,0} \Rightarrow$

$$a = \left[\frac{1}{2} \left(\frac{\Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right) + \frac{1}{2} \left(\frac{\Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right) \right]^{1/2} = \left(\frac{\Omega_{\Lambda,0} - 1}{\Omega_{\Lambda,0}} \right)^{1/2} = a_{\text{bounce}}$$

Radiation + Matter:

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{ro}}{a^4} + \frac{\Omega_{mo}}{a^3}$$

The radiation-matter equality took place when

$$\begin{aligned}\frac{\Omega_{ro}}{a^4} &= \frac{\Omega_{mo}}{a^3} \Rightarrow a = a_{rm} = \frac{\Omega_{ro}}{\Omega_{mo}} \\ &= \frac{9 \times 10^{-5}}{0.31} \approx 2.9 \times 10^{-4}\end{aligned}$$

$$\begin{aligned}H_0 dt &= \frac{da}{\sqrt{\frac{\Omega_{ro}}{a^4} + \frac{\Omega_{mo}}{a^3}}} = \frac{ada}{\sqrt{\Omega_{ro} + \Omega_{mo} \cdot a}} \\ &= \frac{1}{\sqrt{\Omega_{ro}}} \frac{ada}{\sqrt{1 + \frac{\Omega_{mo}}{\Omega_{ro}} \cdot a}} = \frac{1}{\sqrt{\Omega_{ro}}} \frac{ada}{\sqrt{1 + \frac{a}{a_{rm}}}}\end{aligned}$$

$$x = \frac{a}{a_{rm}}, \quad dx = \frac{1}{a_{rm}} da$$

$$\int H_0 dt = \frac{1}{\sqrt{\Omega_{ro}}} a_{rm}^2 \int \frac{x dx}{\sqrt{1+x}}$$

$$H_0 t = \frac{a_{rm}^2}{\sqrt{\Omega_{ro}}} \left. \frac{2}{3} (x-2) \sqrt{1+x} \right|_0^{a/a_{rm}}$$

$$= \frac{2}{3} \frac{a_{rm}^2}{\sqrt{\Omega_{ro}}} \left[\left(\frac{a}{a_{rm}} - 2 \right) \sqrt{1 + \frac{a}{a_{rm}}} - (-2) \right]$$

$$H_0 t = \frac{4}{3} \frac{a_{rm}^2}{\sqrt{\Omega_{ro}}} \left[1 - \left(1 - \frac{a}{2a_{rm}} \right) \left(1 + \frac{a}{a_{rm}} \right)^{1/2} \right]$$

In the limit $a \ll a_{\text{rm}}$:

$$H_{\text{ot}} \approx () \left[1 - \left(1 - \frac{a}{2a_{\text{rm}}} \right) \left(1 + \frac{a}{2a_{\text{rm}}} \right) \right]$$

$$\approx () \left[1 - \left(1 - \frac{a^2}{4a_{\text{rm}}^2} \right) \right]$$

$$H_{\text{ot}} \approx () \frac{1}{4} \frac{a^2}{a_{\text{rm}}^2}$$

$$a(t) \approx \frac{2(H_{\text{ot}})^{1/2} a_{\text{rm}}}{\left(\frac{4}{3} \frac{a_{\text{rm}}^2}{\Omega_{\text{ro}}} \right)^{1/2}} = \left(3 \sqrt{\Omega_{\text{ro}}} H_{\text{ot}} \right)^{1/2} ; (a \ll a_{\text{rm}})$$

In the limit $a \gg a_{\text{rm}}$:

$$H_{\text{ot}} \approx () \left[1 - \left(-\frac{a}{2a_{\text{rm}}} \right) \left(\frac{a}{a_{\text{rm}}} \right)^{1/2} \right]$$

$$\approx () \frac{1}{2} \left(\frac{a}{a_{\text{rm}}} \right)^{3/2}$$

$$\Rightarrow a(t) \approx \left[\frac{2 H_{\text{ot}}}{()} \cdot a_{\text{rm}}^{3/2} \right]^{2/3}$$

$$\approx \left[\frac{2 H_{\text{ot}} a_{\text{rm}}^{3/2}}{\frac{4}{3} \frac{a_{\text{rm}}^2}{\Omega_{\text{ro}}^{1/2}}} \right]^{2/3} \approx \left[\frac{3}{2} \sqrt{\Omega_{\text{ro}}} H_{\text{ot}} \right]^{2/3} ; (a \gg a_{\text{rm}})$$

The time of radiation-matter equality, t_{rm} , can be found by setting $\rho = \rho_{rm}$

$$H_0 t_{rm} = \frac{4}{3} \frac{\Omega_{rm}^2}{\sqrt{\Omega_{r0}}} \left[1 - \left(1 - \frac{1}{\sqrt{2}}\right) (1 + \sqrt{2})^{1/2} \right]$$

$$\left[1 - \frac{1}{2} \sqrt{2} \right]$$

$$\left[1 - \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow t_{rm} = \frac{4}{3} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\Omega_{rm}^2}{\sqrt{\Omega_{r0}}} H_0^{-1}$$

$$= 0.39 \frac{\Omega_{r0}^{3/2}}{\Omega_{m0}^2} H_0^{-1}$$

For $\Omega_{r0} = 9.0 \times 10^{-5}$, $\Omega_{m0} = 0.31$, $H_0^{-1} = 14.4 \text{ Gyr}$

$$\Rightarrow t_{rm} = 3.47 \times 10^{-5} H_0^{-1} \approx 49.968 \times 10^{-6} \times 10^9 \text{ yr}$$

$$\approx 50000 \text{ yr}$$

so, radiation dominance lasted for 50000 yrs, after which universe became matter dominated.

Benchmark Model:

$$k=0$$

Universe contains radiation, matter and cosmological constant

$$H_0 = 68 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

Radiation = photons + neutrinos

$$T_{x0} = 2.7255 \text{ K} = 2.72 - 273.15 = -270.4245^\circ \text{C}$$

$$\frac{T_{y0}}{T_{x0}} = \left(\frac{4}{11}\right)^{1/3} \Rightarrow T_{y0} = 1.95 \text{ K} \quad (\text{for massless neutrinos})$$

The look back time t_L to an object is the difference between the age t_0 of the Universe now (at observation) and the age t_e of the Universe at the time the photons were emitted

$$t_L = t_0 - t_e$$