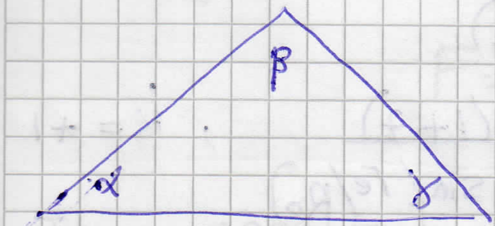
Ch. 4 Cosmic Dynamics

In principle, the radius of curvature of our universe can be determined by measuring the angles α, β, γ of a rather big triangle, and then comparing the sum with

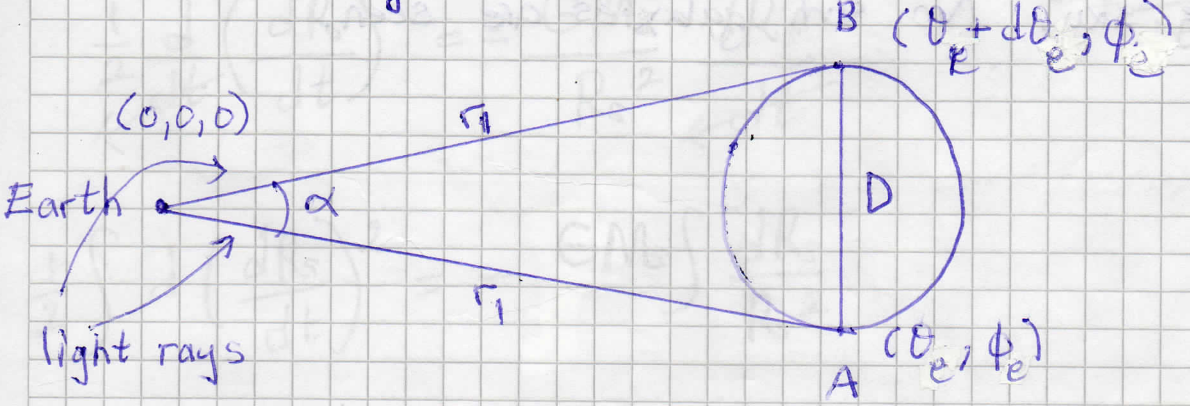
$$\alpha + \beta + \gamma = \pi + \frac{kA}{R_0^2},$$



where A is the area of the triangle.

In practice the deviation of $\alpha + \beta + \gamma$ from π is too small to measure because the largest triangle we can draw has an area A which is much smaller than R_0^2.

If the universe is curved, it can't have a radius of curvature R_0 that is significantly smaller than the current Hubble distance $d_H(t_0) = \frac{c}{H_0} \approx 4380$ Mpc. Let us show why:



According to the Robertson-Walker line element, the proper distance between A and B is obtained by putting $t = t_e =$ const., $\phi = \phi_e =$ const., and $d\theta = \alpha$

$$dl^2 = a(t_e)^2 \left[\underset{=0}{dr^2} + S_k(r_e)^2 \left(d\theta^2 + \underset{1}{\underset{\shortparallel}{\sin^2\theta}}\, \underset{0}{\underset{\shortparallel}{d\phi^2}} \right) \right]$$

$$= a(t_e)^2 S_k(r_e)^2 d\theta^2 = D^2$$

(r is the comoving coordinate, and does not change with time)

$$\Rightarrow \alpha = \frac{D}{a(t_e)S_k(r_e)} = \begin{cases} \dfrac{D}{R_0 \sin(r/R_0)} , & k=+1 \\ D/r , & k=0 \\ \dfrac{D}{R_0 \sinh(\frac{r}{R_0})} , & k=-1 \end{cases}$$

- since $1+z = \dfrac{a(t_0)}{a(t_e)} = \dfrac{1}{a(t_e)}$

$$\Rightarrow \alpha = \frac{D(1+z)}{S_k(r_e)} = \begin{cases} \dfrac{D(1+z)}{R_0 \sin(r_e/R_0)} , & k=+1 \\ \dfrac{D(1+z)}{r_e} , & k=0 \\ \dfrac{D(1+z)}{R_0 \sinh(r_e/R_0)} , & k=-1 \end{cases}$$

Note: For $k=+1$, at $r_e = \pi R_0$, $\alpha = \frac{1}{0} = \infty$ which means at $r = \pi R_0$ the galaxy fills the entire sky. No such galaxies are seen.

The (Newtonian) Friedman Equation:

Consider a homogeneous sphere of matter, with total mass M_s and radius $R_s(t)$. Consider a test mass at the surface of the sphere.

The gravitational force experienced by m is

$$F = -\frac{GM_s m}{R_s(t)^2}$$

The acceleration of m is

$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}$$

Multiply each side by dR_s/dt

$$\frac{dR_s}{dt}\,\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s^2}\cdot\frac{dR_s}{dt} \; ; \; \left(x \equiv dR_s/dt \quad dx^2 = 2x\,dx = 2\frac{dR_s}{dt}\frac{d^2R_s}{dt^2} \right)$$

$$\frac{1}{2}\frac{d}{\cancel{dt}}\left(\frac{dR_s}{dt}\right)^2 = -\frac{GM_s}{R_s^2}\,\frac{dR_s}{\cancel{dt}}$$

$$\frac{1}{2}\int d\left(\frac{dR_s}{dt}\right)^2 = -GM_s\int\frac{dR_s}{R_s^2}$$

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = +\frac{GM_s}{R_s(t)} + \underset{\text{integration constant}}{U} \qquad \circledast$$

$$U = \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 - \frac{GM_s}{R_s(t)} = \frac{K.E}{m} + \frac{P.E}{m} = \text{constant}$$

$M_s = \frac{4\pi}{3}\rho(t) R_s(t)]^3$ = constant mass of the sphere

the scale factor

We can write $R_s(t) = a(t) r_s$

comoving radius of the sphere

Eqn. ⓧ becomes

$$\frac{1}{2} r_s^2 \dot{a}^2 = \frac{4\pi}{3} G r_s^2 \rho(t) a(t)^2 + U$$

Dividing by $\frac{1}{2} r_s^2 a^2 \Rightarrow$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2 a(t)^2}$$

energy density per volume

Now let $kc^2 = -2U$; $\rho(t) \equiv \frac{\varepsilon(t)}{c^2}$; $r_s = R_0$

$$\Rightarrow \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2 a(t)^2}$$

: The Friedmann equation (1922)

$$\downarrow$$

$= H(t)^2$

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{kc^2}{R_0^2} \quad ; (a_0 = 1)$$

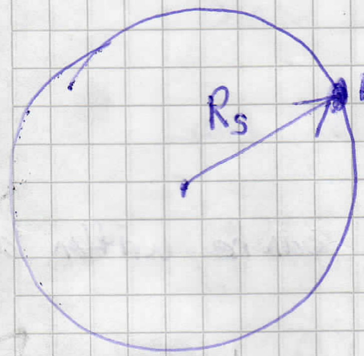
Aleksandr Friedmann

The Newtonian Friedmann Equation:

Ryden Eq. (4.17) $\frac{1}{2} r_s^2 \dot{a}^2 = \frac{4\pi}{3} G r_s^2 \rho(t) a(t)^2 + \frac{E}{m}$

, where U is the gravitational potential (potential energy per unit mass). Dividing by $r_s^2 a^2/2$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2U}{r_s^2 a(t)^2} \qquad (4.18)$$



$R_s(t) = a(t) r_s$

since r_s is time independent

$\frac{k}{r_s} = \frac{k}{R_0}$ = (radius curvature of universe)$^{-1}$

a sphere of radius R_s

Letting $\frac{2U}{r_s^2} = -\frac{kc^2}{R_0^2} \Rightarrow$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R_0^2 a^2}$$

In relativity, not only massive but also massless particles also contribute to the mass density.

Let ε = energy density / volume $= \frac{\rho}{c^2}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2 a(t)^2}$$

$H(t)$: the Hubble parameter

$H(t_0) = H_0$; the Hubble constant

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{kc^2}{R_0^2} \quad ; (a_0 = 1)$$

subscript "0" indicates the present.

In a spatially flat universe ($k=0$) , the Friedmann equation becomes

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon(t)$$

Definition: the critical energy density

$$\varepsilon_c(t) \equiv \frac{3c^2}{8\pi G} H(t)^2$$

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2 = (4870 \pm 290)\ \text{MeV}\ m^{-3}$$

critical mass density $\rho_{c,0} \equiv \dfrac{\varepsilon_{c,0}}{c^2}$

The general Friedmann equation can be written as , upon dividing by $H(t)^2$

$$1 = \frac{8\pi G}{3c^2}\frac{\varepsilon(t)}{H(t)^2} - \frac{kc^2}{R_0^2\, a(t)^2 H(t)^2}$$

$$= \frac{\varepsilon(t)}{\varepsilon_c(t)} - \frac{kc^2}{R_0^2\, a(t)^2 H(t)^2}$$

Definition : the density parameter $\Omega(t)$

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

$$\Rightarrow \quad 1 - \Omega(t) = -\frac{kc^2}{R_0^2\, a(t)^2 H(t)^2}$$

or $$\frac{kc^2}{R_0^2\, a(t)^2 H(t)^2} = \Omega(t) - 1 \qquad (*)$$

Note that :

$*$ if $\Omega(t) = 1 \Rightarrow k = 0$

$*$ if $\Omega(t) > 1 \Rightarrow k = 1$

$*$ if $\Omega(t) < 1 \Rightarrow k = -1$

Since $R_0^2 a(t)^2 H(t)^2 = \left(R_0 a(t) H(t)\right)^2 \doteq +ive$

and does not change sign with time. Therefore LHS of $(*)$ does not change sign, meaning that $\Omega(t) - 1$ also does not change sign with time.

Presently

$$\frac{kc^2}{R_0^2 H_0^2} = \Omega_0 - 1 \quad ; \quad \text{observations}; \quad 0.995 < \Omega_0 < 1.005$$

or $$\frac{k}{R_0^2} = \frac{H_0^2}{c^2}\left(\Omega_0 - 1\right)$$

The Fluid Equation :

The Friedman equation has two unknowns, $a(t)$ and $\mathcal{E}(t)$. We need one more equation to be able to solve for a & $\mathcal{E}$.

1st law of thermodynamics: $dQ = TdS = dE + PdV$, where Q, T, S, E, P, and V are respectively, heat, temperature, entropy, internal energy, pressure and volume.

Since there is no flow of heat into or out of the universe as it expands $dQ = 0$ (adiabatic expansion) and $dS = 0$.

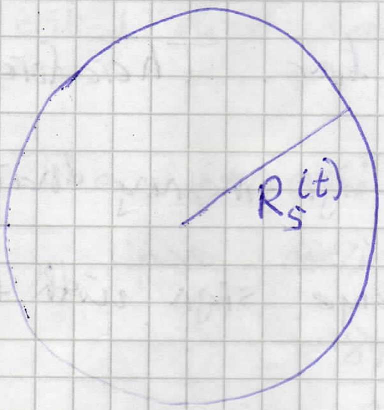
$$\Rightarrow \quad dE + P\,dV = 0$$

Let us take the time derivative

$$\frac{dE}{dt} + \frac{dP}{dt}\cdot dV + P\,\frac{dV}{dt} = 0$$

In a homogeneous universe the density and pressure are everywhere the same, so $\frac{dP}{dt} = 0$

$$\frac{dE}{dt} + P\,\frac{dV}{dt} = 0 \qquad ⊛$$



sphere of radius R_s

$R_s(t) = a(t)\, r_s$: constant
↳ comoving radius

The volume of the sphere is

$$V(t) = \frac{4\pi}{3} R_s^3 = \frac{4\pi}{3} r_s^3\, a(t)^3$$

$$\frac{dV}{dt} = 4\pi r_s^3 a^2 \dot{a} = V\left(3\,\frac{\dot{a}}{a}\right)$$

The internal energy of the sphere is $E(t) = V(t)\,\mathcal{E}(t)$

$$\frac{dE}{dt} = \dot{V}\mathcal{E} + V\dot{\mathcal{E}} = V\left(3\,\frac{\dot{a}}{a}\right)\mathcal{E} + V\dot{\mathcal{E}}$$

$$= V\left(\dot{\mathcal{E}} + 3\,\frac{\dot{a}}{a}\,\mathcal{E}\right) = -P\,\frac{dV}{dt} \quad \leftarrow \text{from } ⊛$$

Then the 1st law thermodynamics in an expanding (or contracting) universe takes the form

$$V\left(\dot{\mathcal{E}} + 3\,\frac{\dot{a}}{a}\,\mathcal{E}\right) + P\,\overbrace{V\,3\,\frac{\dot{a}}{a}}^{dV/dt\ \shortparallel} = 0$$

or

$$\boxed{\dot{\mathcal{E}} + 3\,\frac{\dot{a}}{a}\left(\mathcal{E} + P\right) = 0}$$

The Fluid Equation

In terms of the mass density ρ, the fluid equation takes the form (using $\varepsilon(t) = \rho(t) c^2$)

$$\dot{\rho} c^2 + 3\frac{\dot{a}}{a}\left(\rho c^2 + P\right) = 0$$, dividing by c^2 ⇒

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0}$$

The fluid equation is the same in General Relativity

The acceleration equation:

The Friedman and fluid equations can be combined to an acceleration equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2}$$

× by a^2:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{kc^2}{R_0^2},$$

taking the time derivative

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\varepsilon} a^2 + 2\varepsilon a\dot{a}\right) - 0$$

dividing by $2a\dot{a}$ ⇒

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\varepsilon}\frac{a}{\dot{a}} + 2\varepsilon\right),$$

using the fluid equation $\dot{\varepsilon}\frac{a}{\dot{a}} = -3(\varepsilon + P)$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(-3\varepsilon - 3P + 2\varepsilon\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon + 3P\right)}$$ the acceleration equation

or $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

Note that both positive $\mathcal{E}$ and P cause the acceleration $\ddot{a}$ to be negative $\Rightarrow$ deceleration

A gas of baryonic matter
A gas of photons
A gas of neutrinos
A gas of WIMPs

$\Big\}$ have positive pressure (and $\mathcal{E}$ of course)

So, such matter yields negative acceleration

Suppose there exists a matter with $P < -\frac{1}{3}\mathcal{E}$ then $\mathcal{E} + 3P < \mathcal{E} - \mathcal{E} = 0 \Rightarrow \ddot{a} = +\text{ive}$.

$\Rightarrow$ the expansion of the universe speeds up.

Equations of State:

To solve for $a(t)$, $\mathcal{E}(t)$, and $P(t)$ we need another equation (because the acceleration equation is not an independent equation)

We need an equation of state, namely an equation of the form $P = P(\mathcal{E})$.

We let

$$\underline{P} = \omega \mathcal{E}$$
$$= \omega \rho c^2 ,$$

where ω is a dimensionless number.

As an example, consider a nonrelativistic low density gas, which obeys the ideal gas law

$$PV = nRT = n(k_B N_A)T$$
$$P = \frac{n\mu N_A}{\mu V} k_B T \text{ , where}$$

μ = the mean mass of a gas particle.

$n\mu N_A$ = the total mass of the gas

$\rho = \frac{n \mu N_A}{V}$ = the density of the gas

$$\Rightarrow \quad P = \frac{\rho}{\mu} k_B T \approx \frac{\varepsilon}{\mu c^2} k_B T = \frac{k_B T}{\mu c^2} \varepsilon$$

The Equipartition theorem : $\mu \langle \vec{v}^2 \rangle = 3 k_B T$

$$\Rightarrow \quad P \approx \frac{\varepsilon}{\mu c^2} \frac{1}{3} \mu \langle \vec{v}^2 \rangle = \frac{\varepsilon \langle \vec{v}^2 \rangle}{3c^2}$$

$$P_{nonrel} = \omega \, \varepsilon_{nonrel}, \text{ where } \omega = \frac{\langle \vec{v}^2 \rangle}{3c^2} \ll 1.$$

For example, at room temperature nitrogen molecules have $\langle v \rangle \approx 500 m/s$

$$\omega = \frac{(5 \times 10^2)^2}{3 \times (3 \times 10^8)^2} = \frac{25}{27} \times 10^{-12} \approx 10^{-12}$$

* A gas of photons : $P_{rel} = \frac{\varepsilon_{rel}}{3} \frac{\langle \vec{v}^2 \rangle}{c^2} = \frac{1}{3} \varepsilon_{rel}$, $\omega = \frac{1}{3}$
* Nonrelativistic matter: $P_{nonrel} \approx 0$ $(\omega = 0 \approx 10^{-12})$
* Dark energy : $\omega < -\frac{1}{3}$
* Cosmological constant : $\omega = -1$.

The Cosmological Constant :

If ρ = the mean mass density of the universe

Φ = the gravitational potential

then according to Newtonian gravitation

$$\nabla^2 \Phi = 4\pi G \rho$$

The acceleration of a test particle : $\vec{a} = -\vec{\nabla}\Phi$

In a permenantly static universe $\vec{a}$ must vanish for all the particles everywhere $\Rightarrow \vec{\nabla}\Phi = 0 \Rightarrow \Phi = const.$

$$\Rightarrow \quad \nabla^2 \Phi = 0 = 4\pi G \rho \quad \Rightarrow \quad \rho = 0$$

So, according to this analysis the only permissible static universe is a totally empty universe with $\rho = 0$.
However, if initially, $\rho \neq 0$, and $\dot{a} = \vartheta = 0$ at $t = 0$, then

$\uparrow$ initially static

$$\ddot{a} = \frac{d}{dt}(\dot{a}) = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

$$\int_0^{\dot{a}(t)} d(\dot{a}(t)) = -(\ )\int_0^t dt$$

$$\dot{a}(t) = -(\ )t \quad ; \quad t > 0$$

$\uparrow$ contraction

* if $\rho \neq 0$, and $\dot{a} = \upsilon^{(0)} \neq 0$ (initially expanding)
$= \dot{a}(0) \neq 0$ at $t = 0$

Then,

$$\int_{\dot{a}(0)}^{\dot{a}(t)} d(\dot{a}(t)) = -(\ )t$$

$$\dot{a}(t) - \dot{a}(0) = -(\ )t$$

$$\dot{a}(t) = \dot{a}(0) - (\ )t$$

$$\Rightarrow \text{at} \quad t_* = \frac{\dot{a}(0)}{\frac{4\pi}{3}\left(\rho + \frac{3P}{c^2}\right)}, \quad \dot{a}(t_*) = 0$$

and for $t > t_*$ the universe will collapse
if $\left(\rho + \frac{3P}{c^2}\right) > 0$,

or if $\left(\rho + \frac{3P}{c^2}\right) < 0$ then $\dot{a}(t) > 0$ forever.

These arguments show that a permanently static universe is impossible. (General Relativity predicts the same.)
But Einstein then had a firm belief in a permanently

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

static universe. To obtain a static universe he modified his field equations by adding a Λ-term. In Newtonian terms

$$\nabla^2 \Phi + \underset{\substack{\uparrow}}{\Lambda} = 4\pi G \rho$$

the cosmological constat

If one sets $\Lambda = 4\pi G \rho$ then $\nabla^2 \Phi = 0$ and $\Phi =$ constant and $\ddot{a} = 0$.

The Friedmann Equations with Λ :

In the equation $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{r_s^2 a^2}$; $r_s^2 = R_0^2$

we replace ρ by $\rho_m + \rho_\Lambda$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_\Lambda) - \frac{kc^2}{R_0^2 a^2}$$

$$= \frac{8\pi G}{3} \rho_m + \frac{8\pi G}{3} \rho_\Lambda - \frac{kc^2}{R_0^2 a^2}$$

$\Lambda \rightarrow \Lambda c^2$

in other literature

$\frac{1}{T^2}$ $\sim \frac{1}{L^2}$

$$= \frac{8\pi G}{3} \rho_m + \frac{\Lambda c^2}{3} - \frac{kc^2}{R_0^2 a^2} \; ; \; \Lambda c^2 = 8\pi G \rho_\Lambda$$

The fluid equation :

$\rho = \rho_m + \rho_\Lambda$; $P = P_m + P_\Lambda$

$$\dot{\rho} = \dot{\rho}_m + \cancel{\dot{\rho}_\Lambda}^{\,0} \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \text{ becomes}$$

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}\left(\rho_m + \rho_\Lambda + \frac{P_m}{c^2} + \frac{P_\Lambda}{c^2}\right) = 0$$

For the fluid equation not to change $\rho_\Lambda + \frac{P_\Lambda}{c^2} = 0$

$$\Rightarrow P_\Lambda = -c^2 \rho_\Lambda = -\varepsilon_\Lambda = -\frac{c^2}{8\pi G}\Lambda \; . \; \varepsilon_\Lambda = \frac{\Lambda}{8\pi G}$$

$$\Lambda = \frac{8\pi G}{c^2} \varepsilon_\Lambda = \frac{8\pi G}{c^2} \rho_\Lambda$$

The acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + 3\frac{P_m}{c^2}\right) + \frac{\Lambda}{3}$$

Note that Λ makes $\ddot{a}$ +ive, while ρ_m makes $\ddot{a}$ -ive.

For the universe to remain static, as Einstein desired $\ddot{a} = 0 \Rightarrow$

$$0 = -\frac{4\pi G}{3}(\rho_m + 0) + \frac{\Lambda c^2}{3}$$

$$\Rightarrow \Lambda c^2 = 4\pi G \rho_m$$

Then for $\dot{a} = 0 \Rightarrow$

$$\left(\frac{\dot{a}}{a}\right)^2 = 0 = \frac{8\pi G}{3}\rho_m - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

$$= \frac{8\pi G}{3}\rho_m\left(1 + \frac{1}{2}\right) - \frac{kc^2}{R_0^2 a^2}$$

Today: $a(0) = 1$

$$\frac{kc^2}{R_0^2} = 4\pi G \rho_{m,0} \Rightarrow k = +1$$

$$R_0 = \frac{c}{\sqrt{4\pi G \rho_{m,0}}} = \frac{1}{\sqrt{\Lambda^{1/2}}}$$ (Einstein 1917)

* This Einstein universe is not stable

what is the physical cause of the cosmological constant?

Currently it is the vacuum energy : $\Lambda = \frac{8\pi G}{c^2}\rho_{vac}$

Vacuum is the lowest energy state of a quantum system.

$$\varepsilon_{vac} \sim \frac{E_{Pl}}{l_{Pl}^3} \quad (?)$$

with $E_{Pl} = 1.22 \times 10^{28}$ eV , $l_{Pl} = 1.62 \times 10^{-35}$ m

$\varepsilon_{vac} \sim 3 \times 10^{132} \text{ eV m}^{-3}$

the critical density of the universe

$$\varepsilon_{c,0} \approx 5 \times 10^{3} \times 10^{6} \text{ eV m}^{-3} = 5 \times 10^{9} \text{ eV m}^{-3}$$

today

$$\frac{\varepsilon_{vac}}{\varepsilon_{c,0}} \simeq 10^{123}$$

$$|\Lambda| < 3 \times 10^{-52} \text{ m}^{-2}$$

HW : 4.1 , 4.2 , 4.3 , 4.4 , 4.5

(1st & 2nd edition problems in Ch.4 are the same)

$G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$

$$[\rho_{vac}] = \left[\frac{M}{L^3}\right] = \frac{\text{kg}}{\text{m}^3} \qquad [\Lambda] = \frac{1}{L^2} \sim \frac{1}{\text{m}^2}$$

Note $[\Lambda_{Ryden}] = s^{-2}$

usually $[\Lambda] = m^{-2}$

a

$$\Lambda_R = c^2 \Lambda$$