INTRODUCTION TO COSMOLOGY
EXAM 1

April 1, 2017

Name:

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1. a) State Olber's Paradox.

Why is the night sky dark if the Universe is infinitely large, eternal and static. Instead, it should be bright.

b) What is Olber's explanation?

Distant stars are hidden from view by interstellar matter (dust) that absorbs starlight.

c) What is the correct explanation?

The volume of the presently observable Universe is not infinite, it is in fact too small to contain sufficiently many visible stars.

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2. a) What is the isotropy of the Universe?

There are no preferred directions in the Universe on large scales.

b) What is the homogeneity of the Universe?

There are no preferred locations in the Universe on large scales.

c) What is the cosmological principle?

On large scales, the Universe is homogeneous and isotropic.

3 a) Write down the relation between the proper distance of a certain galaxy from an observer and the scale factor of the universe.

$$d_p(t) = \underbrace{a(t)}_{\text{scale factor}}\ \underbrace{r}_{\text{comoving radial coord.}}$$

b) If the scale factor $a(t) \propto t^{2/3}$, find the comoving coordinate of the galaxy in terms of the present speed of the galaxy and the age of the universe t_0.

Hubble's law: $v(t) = H(t)\,d(t) = H(t)\,a(t)\,r = \frac{\dot a}{a}.a.r$

$= \dot a r$; $a(t) = (t/t_0)^{2/3}$

$$\dot a = \frac{2}{3}\frac{t^{-1/3}}{t_0^{2/3}} \Rightarrow r = \frac{v(t_0)}{\dot a(t_0)} = \frac{v(t_0)}{\frac{2}{3}\frac{1}{t_0}} = \frac{3t_0 v(t_0)}{2}$$

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4. a) Write down the expression for the Robertson-Walker metric.

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_k(r)^2 d\Omega^2\right],$$

where $S_k(r) = \begin{cases} R\sin(r/R) ; & k=+1 \\ r ; & k=0 \\ R\sinh(r/R) ; & k=-1 \end{cases}$; R = radius of curvat.

b) How many space dimensions are there in the R-W metric?

Three; (r, θ, ϕ)

c) What is the dimension of the embedding space of the universe?

Four

d) Is the embedding space curved or flat?

Flat

5. a) The surface of the Earth can be idealized as a perfect sphere of radius R. Find the circumference of a circle of radius r.

$$dl^2 = dr^2 + R^2 \sin^2(r/R)\, d\theta^2$$

$$C = \int dl = \int \sqrt{\underset{0}{\underset{\parallel}{dr^2}} + R^2 \sin^2(r/R)\, d\theta^2} = R\sin(r/R)\int_0^{2\pi} d\theta$$

$$C = 2\pi R \sin(r/R)$$

b) What would be the circumference of a circle of radius r if the surface of the Earth were flat?

$$C = 2\pi r$$

c) Find the expression for the deviation from flatness.
(Use $\sin x \approx x - \frac{x^3}{3!}$ for small x.)

$$\Delta C = 2\pi r - 2\pi R \sin(r/R) = 2\pi\left(r - R\left(\frac{r}{R} - \frac{1}{3!}\frac{r^3}{R^3}\right)\right)$$

$$= 2\pi \cdot \frac{1}{6}\frac{r^3}{R^2} = \frac{\pi}{3}\frac{r^3}{R^2}$$

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6. a) Write down the Friedmann equation with the cosmological constant Λ.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

b) Assume that Λ is not a true constant, but varies with time. What must be the expression for $\Lambda(a)$ so that the Universe mimics a flat universe with no cosmological constant.

We want $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\varepsilon$, which is possible

if $-\frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3} = 0$

$$\Rightarrow \Lambda(a) = \frac{3k}{R_0^2 a^2}$$

c) Assume that the energy density $\mathcal{E}$ is proportional to a^{-1}. Then find the time dependence of the scale factor $a(t)$.

$$\mathcal{E} = \mathcal{E}_0 a^{-1} \; ; \quad \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2}\mathcal{E}_0 \frac{1}{a}$$

$$\frac{da}{dt} = \left(\frac{8\pi G \mathcal{E}_0}{3c^2}\right)^{1/2} a^{-1/2} \Rightarrow \int_0^a da\, a^{1/2} = (\quad)^{1/2}\int_0^t dt$$

$$2a^{1/2} = \left(\frac{8\pi G}{3c^2}\mathcal{E}_0\right)^{1/2} \cdot t \Rightarrow a(t) = \left(\frac{2\pi G}{3c^2}\mathcal{E}_0\right) t^2 = \left(\frac{t}{t_0}\right)^2,$$

, where $t_0 = \left(\frac{2\pi G \mathcal{E}_0}{3c^2}\right)^{-1/2}$

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7. a) Write down the acceleration equation.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\mathcal{E} + 3P)$$

b) If the universe is accelerating as it expands, what must be the value of ω? ($P = \omega \mathcal{E}$)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\mathcal{E}(1+3\omega) > 0 \Rightarrow (1+3\omega) < 0$$

$$\Rightarrow \omega < -\frac{1}{3}$$

c) What kind of a component of the universe can have such an ω value?

A cosmological constant

8. Assume that the scale factor of the universe is given by $a(t) = (t/t_0)^2$, where t_0 is the present time.

a) The light emitted by a galaxy at $t = t_e$ reaches us today. Calculate the proper distance of that galaxy.

From $ds^2 = -c^2 dt^2 + a^2 dr^2 = 0$

$$dr = c\frac{dt}{a}$$

$$d_p(t_0) = \int dr = c\int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$d_p(t_0) = c\,t_0^2 \int_{t_e}^{t_0} t^{-2}\,dt = c\,t_0^2 \left.\frac{t^{-1}}{(-1)}\right|_{t_e}^{t_0}$$

$$= c\,t_0^2\left[-\frac{1}{t_0} + \frac{1}{t_e}\right] = c\,t_0\left(\frac{t_0}{t_e} - 1\right)$$

b) Express the proper distance in terms of the redshift z.

$$1+z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = \frac{t_0^2}{t_e^2} \Rightarrow \frac{t_0}{t_e} = \sqrt{1+z}$$

$$\Rightarrow d_p(t_0) = c\,t_0\left(\sqrt{1+z} - 1\right)$$

c) Find the proper distance of that galaxy at the time of light emission t_e.

From $$\frac{d_p(t_0)}{d_p(t_e)} = \frac{\int a(t_0)\,dr}{\int a(t_e)\,dr} = \frac{a(t_0)\int dr}{a(t_e)\int dr} = \frac{1}{a(t_e)} = 1+z$$

$$\Rightarrow d_p(t_e) = \frac{d_p(t_0)}{1+z} = c\,t_0\left(\frac{\sqrt{1+z} - 1}{1+z}\right)$$

9. Consider a universe with $a(t) = (t/t_0)^{1/2}$.

a) Calculate the horizon distance $d_{hor}(t)$ at time t.

$$d_{hor}(t) = c\int_0^t \frac{dt}{a(t)} = c\,t_0^{1/2}\int_0^t t^{-1/2}dt$$

$$= c\,t_0^{1/2}.2t^{1/2} = 2c\,t_0^{1/2}.\,t^{1/2}$$

b) Find the horizon distance at $t = t_0$ (today).

$$d_{hor}(t_0) = 2ct_0$$

c) Find the Hubble distance $d_H(t) = cH(t)^{-1}$, where $H(t)$ is the Hubble parameter, and c is the speed of light

$$d_H(t) = cH(t)^{-1} = c\left(\frac{\dot{a}}{a}\right)^{-1} = c\left[\frac{\frac{1}{2}t^{-1/2}}{t^{1/2}}\right]^{-1} = c\,2t$$

d) Find the ratio $\dfrac{d_H(t_0)}{d_{hor}(t_0)}$.

$$\frac{d_H(t_0)}{d_{hor}(t_0)} = \frac{2ct_0}{2ct_0} = 1.$$

10. Suppose the equation of state is $P = (\gamma - 1)\mathcal{E}$, where γ is a constant in the range $0 < \gamma < 2$. Assuming $k = 0$ and $\Lambda = 0$,

a) Obtain $\mathcal{E}(a)$ using the fluid equation.

The fluid equation: $\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E} + P) = 0$

$$\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}\mathcal{E}(1 + \gamma - 1) = 0$$

$$\frac{1}{\mathcal{E}}\frac{d\mathcal{E}}{dt} + \frac{3}{a}\frac{da}{dt}(\gamma) = 0$$

$$\int \frac{d\mathcal{E}}{\mathcal{E}} = -3\gamma \int \frac{da}{a}$$

$$\ln \mathcal{E} = -3\gamma \ln a + \text{const.}$$

$$\ln \mathcal{E} = \ln a^{-3\gamma} + \text{const.}$$

$$\Rightarrow \mathcal{E} = \exp\left[\ln a^{-3\gamma} + \text{Const.}\right]$$

$$= e^{\text{const.}} a^{-3\gamma}$$

$$\mathcal{E}(a) = \mathcal{E}_0 a^{-3\gamma},$$

where $\mathcal{E}_0 = e^{\text{const.}}$

b) Obtain $a(t)$ using the Friedmann equation.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\mathcal{E}$$

$$= \frac{8\pi G}{3c^2}\mathcal{E}_0 a^{-3\gamma}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\mathcal{E}_0 a^{2-3\gamma}$$

$$\dot{a} = \left(\frac{8\pi G}{3c^2}\mathcal{E}_0\right)^{1/2} a^{1-3\gamma/2}$$

$$\int_0^a a^{-1+3\gamma/2}\, da = (\quad)^{1/2}\int_0^t dt$$

$$\frac{a^{3\gamma/2}}{3\gamma/2} = (\quad)^{1/2} t$$

$$a = \left(\frac{3\gamma}{2}\right)^{\frac{2}{3\gamma}} \left(\frac{8\pi G\mathcal{E}_0}{3c^2}\right)^{\frac{1}{3\gamma}} t^{2/3\gamma}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3\gamma}$$

Check: For $\gamma = 1$

$P = 0 \Rightarrow w = 0$

$$\Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

, which is correct.

INTRODUCTION TO COSMOLOGY

EXAM 1

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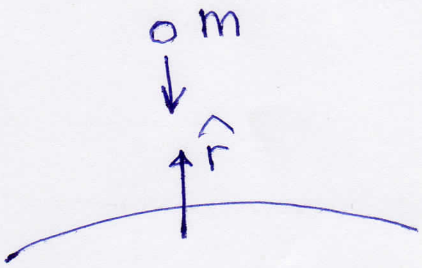
April 14, 2018

Student:

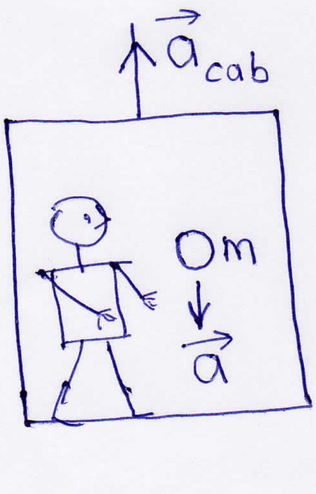
1a. Consider a particle of inertial mass m_i and gravitational mass m_g. When the particle is dropped from rest at a height h from the surface of the earth, what is the acceleration of the particle?



$$\vec{F} = m_i \vec{a} = m_g \vec{g} = m_g g(-\hat{r})$$

$$\Rightarrow \vec{a} = \frac{m_g}{m_i} g(-\hat{r})$$

b) Now suppose that the particle is taken to deep space where there are no fields of any kind. If the particle is dropped by an observer in a cabin in deep space and observed that it has the same value of the acceleration in part a. What must be the acceleration of the cabin? (Do not forget to use unit vectors for the direction of acceleration.)



The equivalence of gravity and inertia requires $\vec{a}$ to be as in part a).

Then $\vec{a}_{cab} = -\vec{a} = \frac{m_g}{m_i} g \hat{r}$ upward

2\. a) Write down the expression for the Robertson–Walker metric.

$$ds^2 = -c^2 dt^2 + a(t)^2\left[dr^2 + S_k^2 \overbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}^{d\Omega^2}\right]$$

$$S_k(r) = \begin{cases} R_0 \sin\left(\frac{r}{R_0}\right); & k=1 \\ r; & k=0 \\ R_0 \sinh\left(\frac{r}{R_0}\right); & k=-1 \end{cases}$$

b) Find the distance traveled by a light ray emitted at $t=0$ and observed at t.

$d(t) = a(t) r$; r is found as follows: For light rays travelling radially $ds=0$ and $d\Omega = 0$ $\Rightarrow -c^2 dt^2 + a(t)^2 dr^2 = 0$

$\Rightarrow r = c\int_0^t \frac{dt}{a(t)}$ $\Rightarrow d(t) = c\, a(t) \int_0^t \frac{dt}{a(t)}$.

c) What is this distance called?

Horizon distance

3\. a) Consider a universe whose energy density is equal to the critical energy density and the energy density is contributed by a cosmological constant.

Solve the Friedmann equation and obtain the scale factor $a(t)$.

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\mathcal{E}_c = \frac{\Lambda c^2}{3}; \quad \mathcal{E}_c = \mathcal{E}_\Lambda = \frac{c^4}{8\pi G}\Lambda$$

$$\Rightarrow \frac{\dot a}{a} = \frac{da}{a}\frac{1}{dt} = \left(\frac{\Lambda c^2}{3}\right)^{1/2}$$

$$\int \frac{da}{a} = \left(\frac{\Lambda c^2}{3}\right)^{1/2}\int dt$$

$$\ln a = \left(\frac{\Lambda c^2}{3}\right)^{1/2} t + \text{const.}$$

$$\Rightarrow a(t) = \exp\left[\sqrt{\frac{\Lambda c^2}{3}}\, t + \text{const.}\right] = e^{\text{const.}}\, e^{\sqrt{\frac{\Lambda c^2}{3}}\, t} = \text{Const.}\, e^{H_0 t},$$

$$H_0 = \sqrt{\frac{\Lambda c^2}{3}} = \sqrt{\frac{8\pi G \mathcal{E}_\Lambda}{3c^2}}$$

b) What is the value of the density parameter $\Omega(t)$ for this model universe?

$$\Omega = \frac{\mathcal{E}_\Lambda}{\mathcal{E}_c} = \frac{\mathcal{E}_c}{\mathcal{E}_c} = 1$$

c) What is the value of the curvature parameter k?

$$k = 0$$

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4. Consider that dark energy may be due to an exotic fluid with an equation of state $P_{de} = -\frac{5}{6}\rho_{de} c^2$ that does not interact with anything else.
If a homogeneous spatially-flat universe with scale factor a contains only this form of dark energy and pressureless matter with density ρ_m:

a) Use the fluid equation for a noninteracting fluid and find ρ_{de} as a function of a.
(Take $a_0 = 1$ today

$$\dot{\rho}_{de} + 3\frac{\dot{a}}{a}\left(\rho_{de} + \frac{P_{de}}{c^2}\right) = 0$$

$$\dot{\rho}_{de} + 3\frac{\dot{a}}{a}\rho_{de}\left(1 - \frac{5}{6}\right) = 0$$

$$\frac{\dot{\rho}_{de}}{\rho_{de}} + \frac{1}{2}\frac{\dot{a}}{a} = 0$$

$$\frac{1}{\cancel{dt}}\frac{d\rho_{de}}{\rho_{de}} = -\frac{1}{2}\frac{da}{a}\frac{1}{\cancel{dt}}$$

$$\int \frac{d\rho_{de}}{\rho_{de}} = -\frac{1}{2}\int \frac{da}{a}$$

$$\ln \rho_{de} = -\frac{1}{2}\ln a + \text{const.}$$

$$= \ln a^{-1/2} + \text{const.}$$

$$\Rightarrow \rho_{de} = e^{\text{const.}} \cdot a^{-1/2}$$

$$\rho_{de}(a) = \frac{C_0}{\sqrt{a}}$$

b) Use the fluid equation for a noninteracting fluid and find ρ_m as a function of a.

For pressureless matter

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}\rho_m = 0$$

$$\frac{\dot{\rho}_m}{\rho_m} = -3\frac{\dot{a}}{a}$$

$$\int \frac{d\rho_m}{\rho_m} = -3\int \frac{da}{a}$$

$$\ln\rho_m = -3\ln a + \text{const.}$$

$$\Rightarrow \rho_m = \frac{\text{Const.}}{a^3} \Rightarrow \text{Const.} = \rho_{m,0}\cdot \overset{1}{\overset{\shortparallel}{a(t_0)}} = \rho_{m,0}$$

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5. When a spectral line from a galaxy is observed at wavelength λ_o, we know that long ago it was emitted at wavelength λ_e.

a) What is the definition of the redshift z?

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\lambda_o}{\lambda_e} - 1$$

b) Assuming that wavelength is proportional to scale factor $a(t)$, express the redshift in terms of $a(t_0)$ and $a(t_e)$.

From a) $\Rightarrow$ $$1+z = \frac{\lambda_o}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} \; ; \; (a(t_0) = 1)$$

c) Today the universe is assumed to consists mainly of a cosmological constant Λ and matter. What is the equation of state ($p = w\rho c^2$) for Λ?

$$p = -\rho c^2$$

d) If $\Omega_{\Lambda,0} = 0.7$, show that $\frac{\rho_m}{\rho_{m,0}} = \frac{7}{3}\frac{\rho_m}{\rho_{\Lambda,0}}$.

$\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{\rho_{c,0}} = 0.7$

$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{c,0}} = 0.3$

$$\frac{\rho_m}{\rho_{m,0}} = \frac{\rho_m}{0.3\rho_{c,0}} = \frac{\rho_m}{0.3(\rho_{\Lambda,0}/0.7)} = \frac{7}{3}\frac{\rho_m}{\rho_{\Lambda,0}}$$

e) What was $\frac{\rho_m}{\rho_\Lambda}$ at redshif 2 ?

$\rho_m = \frac{\rho_{m,0}}{a^3}$

$\rho_\Lambda = \text{const.} = \rho_{\Lambda,0}$

$$\frac{\rho_m}{\rho_{m,0}} = \frac{1}{a^3} = (1+z)^2 = (1+2)^2 = 9$$

$$\frac{\rho_m}{\rho_{\Lambda,0}} = \frac{\rho_m}{\rho_\Lambda} = \frac{3}{7}\frac{\rho_m}{\rho_{m,0}} = \frac{3}{7}\times 9 = \frac{27}{7}$$

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6. Consider a universe with scale factor $a(t) = \left(\frac{t}{t_0}\right)^n$, where $n > 0$.

a) Calculate the horizon distance $d_{hor}(t)$ at time t.

$$d_{hor}(t) = ca\int_0^t \frac{dt}{a} = c\left(\frac{t}{t_0}\right)^n t_0^n \int_0^t t^{-n}dt = ct^n\frac{t^{-n+1}}{1-n} = \frac{ct}{1-n}$$

b) Find the Hubble distance $d_H(t) = \frac{c}{H(t)}$.

$$H(t) = \frac{\dot{a}}{a} = \frac{nt^{n-1}}{t^n} = nt^{-1}$$

$$\Rightarrow d_H(t) = \frac{ct}{n}$$

c) Find the value of n so that $d_{hor}(t_0) = d_H(t_0)$.

$$d_{hor}(t) = d_H(t) \Rightarrow \frac{ct}{1-n} = \frac{ct}{n} \Rightarrow \frac{1}{1-n} = \frac{1}{n}$$

$$1-n = n \Rightarrow \boxed{n = \frac{1}{2}}$$

7 a) Write down the Friedmann equation with a cosmological constant Λ.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

b) Assume that $\rho = \frac{C_1}{a^2} + C_2$, where C_1 and C_2 are positive constants. Determine C_1 and C_2 such that the right hand side of the Friedmann equation equals Λc^2.

$$\text{RHS} = \frac{8\pi G}{3}\frac{C_1}{a^2} + \frac{8\pi G}{3}C_2 - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3} = \Lambda c^2$$

$$\left(\frac{8\pi G}{3}C_1 - \frac{kc^2}{R_0^2}\right)\frac{1}{a^2} = 0 \Rightarrow C_1 = \frac{3kc^2}{8\pi G R_0^2}$$

$$\frac{8\pi G}{3}C_2 + \frac{\Lambda c^2}{3} = \Lambda c^2 \Rightarrow C_2 = \frac{3}{8\pi G}\left(\frac{2}{3}\Lambda c^2\right) = \frac{\Lambda c^2}{4\pi G}$$

c) Then find $a(t)$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda c^2$$

$$\frac{\dot{a}}{a} = \Lambda^{1/2} . c$$

$$\int \frac{da}{a} = c\Lambda^{1/2}\int dt$$

$$\ln a = c\Lambda^{1/2} t + \text{const.}$$

$$a(t) = e^{\text{const.}} e^{c\Lambda^{1/2} t}$$

$$a(t) = \text{Const.}\, e^{c\sqrt{\Lambda}\, t}$$

8. a) Consider a sphere of radius $R_s(t)$ of the universe as it expands adiabatically. From the conservation of energy equation $dQ = dE + PdV$ obtain the fluid equation involving $\mathcal{E}$ (energy density), P (pressure), and a (scale factor).

$V = \text{const.}\, a^3$

$dQ = 0$

$E = \mathcal{E} V = \text{const.}\, \mathcal{E} a^3$

$dE + PdV = 0$

$d(\mathcal{E} a^3) + P\, da^3 = 0$

$a^3 d\mathcal{E} + 3a^2 da\, \mathcal{E} + 3Pa^2 da = 0$

dividing by a^3:

$d\mathcal{E} + 3\frac{da}{a}\mathcal{E} + 3P\frac{da}{a} = 0$

Taking the time derivative

$\frac{d\mathcal{E}}{dt} + 3\frac{1}{a}\frac{da}{dt}(\mathcal{E}+P) = 0 \Rightarrow \boxed{\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E}+P) = 0}$

b) Obtain the "acceleration equation" from the Friedmann equation and the fluid equation.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

x by a^2:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\mathcal{E} a^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}(\dot{\mathcal{E}} a^2 + 2\mathcal{E} a\dot{a})$$

Dividing by $2a\dot{a}$:

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\mathcal{E}}\frac{a}{\dot{a}} + 2\mathcal{E}\right)$$

using the fluid eq.

$$\dot{\mathcal{E}}\frac{a}{\dot{a}} = -3(\mathcal{E}+P) \Rightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(-3\mathcal{E} - 3P + 2\mathcal{E})$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\mathcal{E}+3P)}$$

The acceleration equation.

c) If the universe is accelerating as it expands, what must be the value of γ defined by $P = (\gamma-1)\mathcal{E}$?

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\mathcal{E}\underbrace{(1+3\gamma-3)}_{(3\gamma-2)}$$

$$= +\frac{4\pi G}{c^2}\mathcal{E}\left(\frac{2}{3}-\gamma\right) > 0$$

$$\frac{2}{3} - \gamma > 0$$

$$\boxed{\gamma < \frac{2}{3}}$$

9. a) Define the density parameter Ω.

$$\Omega = \frac{\rho}{\rho_c} = \frac{\mathcal{E}}{\mathcal{E}_c}$$, where $\mathcal{E}_c$ is the critical energy density.

b) Using the Friedmann equation find $\frac{1-\Omega}{\Omega}$.

$$H^2 = \underbrace{\frac{8\pi G}{3c^2}}_{\mathcal{E}_c^{-1}} \mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

$$1 = \frac{8\pi G}{3c^2}\frac{\mathcal{E}}{H^2} - \frac{kc^2}{R_0^2 a^2 H^2}$$

$$1 = \underbrace{\frac{\mathcal{E}}{\mathcal{E}_c}}_{\Omega} - \frac{kc^2}{R_0^2 a^2 H^2}$$

$$1 - \Omega = -\frac{kc^2}{R_0^2 a^2 H^2}$$

$$\frac{1-\Omega}{\Omega} = -\frac{kc^2}{R_0^2 a^2 H^2} \cdot \frac{\mathcal{E}_c}{\mathcal{E}}$$

$$\boxed{\frac{1-\Omega}{\Omega} = -\frac{3kc^4}{R_0^2 a^2 8\pi G}\frac{1}{\mathcal{E}}}$$

c) Using b), state the conditions on Ω for closed, flat and open universes.

Closed universe: $k = +1 \Rightarrow \Omega > 1$

Flat universe: $k = 0 \Rightarrow \Omega = 0$

Open universe: $k = -1 \Rightarrow \Omega < 1$

d) How does $\mathcal{E}$ (or ρ) vary with the scale factor a in the standard model of cosmology when the universe is matter dominated?

$$\mathcal{E} = \frac{\mathcal{E}_{m,0}}{a^3}$$

e) Then find how $\frac{1-\Omega}{\Omega}$ varies with a in the matter dominated era.

$$\frac{1-\Omega}{\Omega} = -\left(\quad\right)\frac{1}{a^2}\frac{a^3}{\mathcal{E}_{m,0}} = -\text{const.}\, a$$

10. Consider the definition of the redshift in question 5b again.

a) What is the redshift today?

$$1+z = \frac{a(t_0)}{a(t_e)} \quad ; \text{ today } a(t_e) = a(t_0)$$

$$\Rightarrow \quad 1+z = 1 \quad \Rightarrow \boxed{z=0 \text{ today}}$$

b) What was the redshift when the universe was $\frac{1}{10}$ of its present size?

$$a(t_e) = \frac{1}{10}\, a(t_0)$$

$$1+z = \frac{a(t_0)}{\frac{1}{10}\, a(t_0)} = 10$$

$$\Rightarrow \boxed{z=9}$$