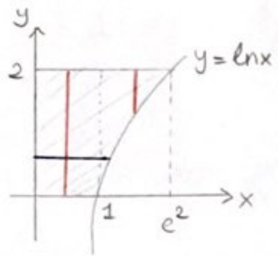
MAT1072 MATHEMATICS II EXERCISES

① Calculate the area of the region enclosed by the curve $y=\ln x$ and the lines $y=0$, $y=2$ and by the y-axis.



$$A=\int_{y=a}^{y=b}\int_{x=g_1(y)}^{x=g_2(y)} dx\,dy=\int_{x=c}^{x=d}\int_{y=h_1(x)}^{y=h_2(x)} dy\,dx$$

First way: Use the horizontal "blue" line —

Region : $x=0$ to $x=e^y$
$y=0$ to $y=2$

$$A=\int_0^2\int_0^{e^y} dx\,dy=\int_0^2 x\Big|_0^{e^y} dy=\int_0^2 e^y dy=e^y\Big|_0^2=e^2-1$$

Second way: Use the vertical "red" lines |

Region $\left\{\begin{matrix} y=0 \text{ to } 2 \\ x=0 \text{ to } 1\end{matrix}\right\}$ and $\left\{\begin{matrix} y=\ln x \text{ to } 2 \\ x=1 \text{ to } e^2\end{matrix}\right\}$

$$A=\int_0^1\int_0^2 dy\,dx+\int_1^{e^2}\int_{\ln x}^{2} dy\,dx=2+\int_1^{e^2}(2-\ln x)\,dx$$

$$=2+(2-\ln x).x-\int_1^{e^2}(-1)\,dx=2+\left(x(2-\ln x)+x\right)\Big|_1^{e^2}$$

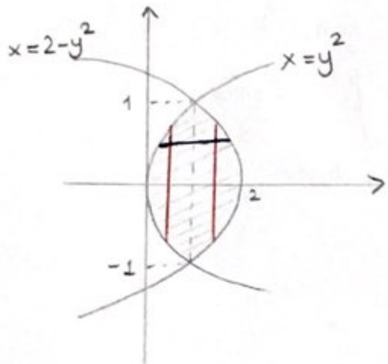
$2-\ln x=u$, $dx=dv$
$-\frac{dx}{x}=du$, $v=x$

$$=2+\left(e^2(2-2)+e^2-2-1\right)$$

$$=e^2-1$$

② Calculate the integral of the function $f(x,y) = 1+5y$ on the region R which is enclosed by the curves $x=y^2$ and $x=2-y^2$

First way:



$$\left.\begin{array}{l} x=y^2 \\ x=2-y^2 \end{array}\right\} \; y^2=2-y^2 \Rightarrow y=\mp 1$$

Region for horizontal line —

$y=-1$ to $y=1$

$x=y^2$ to $x=2-y^2$

$$A=\iint_R (1+5y)\,dA = \int_{-1}^{1}\int_{y^2}^{2-y^2}(1+5y)\,dx\,dy = \int_{-1}^{1}(x+5xy)\Big|_{y^2}^{2-y^2}dy$$

$$= \int_{-1}^{1}(2-2y^2+10y-10y^3)\,dy = \left(2y-\frac{2}{3}y^3+5y^2-\frac{5}{2}y^4\right)\Big|_{-1}^{1} = \frac{8}{3}$$

Second way: Regions for vertical lines |

Region 1 : $x=0$ to $x=1$

$y=-\sqrt{x}$ to $y=\sqrt{x}$

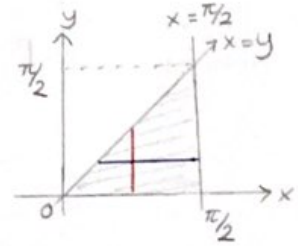
Region 2 : $x=1$ to $x=2$

$y=-\sqrt{2-x}$ to $y=\sqrt{2-x}$

$$A=\iint_R (1+5y)\,dA = \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}(1+5y)\,dy\,dx + \int_1^2\int_{-\sqrt{2-x}}^{\sqrt{2-x}}(1+5y)\,dy\,dx$$

③ Sketch the region of the integration, reverse the order of the integration and evaluate the integral.

a) $\displaystyle\int_0^{\pi/2}\int_y^{\pi/2} \frac{\sin x}{x}\,dx\,dy$



Region according to horizontal line —

$y=0$ to $y=\pi/2$
$x=y$ to $x=\pi/2$

Region according to vertical line |

$x=0$ to $x=\pi/2$
$y=0$ to $y=x$

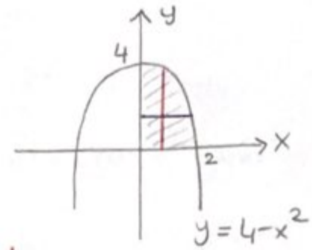
So, $\displaystyle\int_0^{\pi/2}\int_y^{\pi/2} \frac{\sin x}{x}\,dx\,dy = \int_0^{\pi/2}\int_0^{x} \frac{\sin x}{x}\,dy\,dx$

$$= \int_0^{\pi/2} y\,\frac{\sin x}{x}\Big|_0^x\,dx$$

$$= \int_0^{\pi/2} \sin x\,dx = -\cos x\Big|_0^{\pi/2}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right) = \underline{1}.$$

b) $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{x.e^{2y}}{4-y}\,dy\,dx$



$\left.\begin{array}{l} x=0 \text{ to } 2 \\ y=0 \text{ to } 4-x^2 \end{array}\right\}$ vertical region |

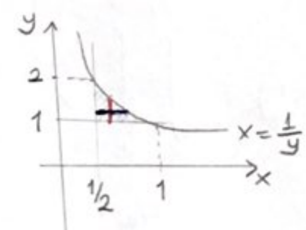
Then,

$\left.\begin{array}{l} y=0 \text{ to } 4 \\ x=0 \text{ to } \sqrt{4-y} \end{array}\right\}$ horizontal region —

So, $\displaystyle A = \int_0^2 \int_0^{4-x^2} \frac{x.e^{2y}}{4-y}\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x.e^{2y}}{4-y}\,dx\,dy$

$$= \int_0^4 \frac{x^2}{2}\cdot\frac{e^{2y}}{4-y}\Bigg|_0^{\sqrt{4-y}} dx = \int_0^4 \frac{e^{2y}}{2}\,dx = \frac{e^{2y}}{4}\Bigg|_0^4 = \frac{e^8-1}{4}$$

c) $\displaystyle\int_1^2 \int_{1/2}^{1/y} e^{\ln x - x}\,dx\,dy$



$\left.\begin{array}{l} y=1 \text{ to } 2 \\ x=1/2 \text{ to } 1/y \end{array}\right\}$ Horizontal Region — Then, $\left.\begin{array}{l} x=1/2 \text{ to } x=1 \\ y=1 \text{ to } y=1/x \end{array}\right\}$ Vertical Region |

$$\Rightarrow \int_1^2 \int_{1/2}^{1/y} e^{\ln x - x}\,dx\,dy = \int_{1/2}^1 \int_1^{1/x} e^{\ln x - x}\,dy\,dx = \int_{1/2}^1 y e^{\ln x - x}\Big|_1^{1/x} dx = \int_{1/2}^1 \left(\frac{1}{x}-1\right)e^{\ln x - x}\,dx$$

$u = \ln x - x$
$du = \frac{1}{x} - 1$

$$= \int e^u\,du = e^{\ln x - x}\Big|_{1/2}^1 = e^{-1} - \frac{1}{2}e^{-1/2} = \frac{1}{e} - \frac{1}{2\sqrt{e}}$$

④ Evaluate the double integral $\iint_D \frac{y}{x^5+1}\, dA$,

where $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$

$$\iint_D \frac{y}{x^5+1}\, dA = \int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx = \int_0^1 \left(\frac{y^2}{2(x^5+1)} \right)_0^{x^2} dx$$

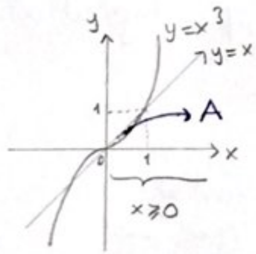
$$= \int_0^1 \left(\frac{(x^2)^2}{2(x^5+1)} - 0 \right) dx = \int_0^1 \frac{x^4}{2(x^5+1)}\, dx$$

$x^5+1 = u$
$5x^4 dx = du$

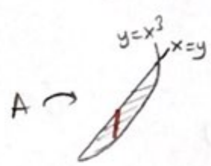
$$= \int_1^2 \frac{1}{10u}\, du = \frac{1}{10} \ln u \Big|_1^2 = \frac{\ln 2}{10} - \frac{\ln 1}{10} = \frac{\ln 2}{10}$$

⑤ Find the integral of the function $f(x,y) = x^2 + 2y$, on the region bounded by $y = x$, $y = x^3$ and $x \geqslant 0$.

$x = x^3 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2-1) = 0 \Rightarrow x = 1$ and $x = 0$ since $x \geqslant 0$.



$$A = \iint_D (x^2+2y)\, dA = \int_0^1 \int_{x^3}^{x} (x^2+2y)\, dy\, dx$$



$$= \int_0^1 \left(x^2 y + y^2 \right)_{x^3}^{x} dx = \int_0^1 \left(x^3 + x^2 - x^6 - x^5 \right) dx$$

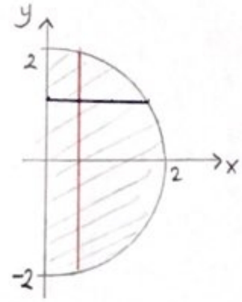
$$= \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^7}{7} - \frac{x^6}{6} \Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{1}{7} - \frac{1}{6}$$

Vertical region:
$x = 0$ to 1
$y = x^3$ to x

$$= \frac{23}{84}$$

⑥ Sketch the region of integration and change the order of integration.

$$R = \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$



$x = \sqrt{4-y^2} \Rightarrow x^2 = 4-y^2 \Rightarrow x^2+y^2 = 4$ ↴ circle with r=2.

$\left.\begin{array}{l} y = -2 \text{ to } 2 \\ x = 0 \text{ to } \sqrt{4-y^2} \end{array}\right\}$ Horizontal Region —

So, $\left.\begin{array}{l} x = 0 \text{ to } 2 \\ y = -\sqrt{4-x^2} \text{ to } \sqrt{4-x^2} \end{array}\right\}$ vertical Region |

Thus, $$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy = \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

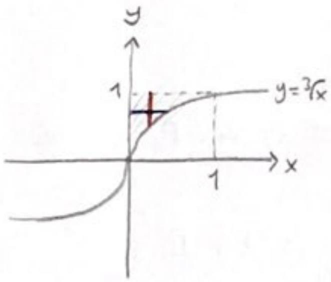
⑦ Evaluate $\int_0^1\int_x^1 e^{\frac{x}{y}}\,dy\,dx$ by reversing the order of integration.

$\left.\begin{array}{l} x = 0 \text{ to } 1 \\ y = x \text{ to } 1 \end{array}\right\}$ Vertical line | $\Rightarrow$ $\left.\begin{array}{l} y = 0 \text{ to } 1 \\ x = 0 \text{ to } y \end{array}\right\}$ Horizontal line —

$$\Rightarrow \int_0^1\int_x^1 e^{x/y}\,dy\,dx = \int_0^1\int_0^y e^{x/y}\,dx\,dy = \int_0^1 y\cdot e^{x/y}\Big|_0^y\,dy$$

$$= \int_0^1 (ye - y)\,dy = \frac{y^2}{2}(e-1)\Big|_0^1 = \frac{e-1}{2}$$

⑧ Evaluate the integral $\int_0^1 \int_{\sqrt[3]{x}}^1 e^{y^4}\,dy\,dx$.



$$\left.\begin{array}{l} x = 0 \text{ to } 1 \\ y = \sqrt[3]{x} \text{ to } 1 \end{array}\right\} \text{ Vertical line } |$$

$$\Rightarrow \left.\begin{array}{l} y = 0 \text{ to } 1 \\ x = 0 \text{ to } y^3 \end{array}\right\} \text{ Horizontal line } —$$

$$\Rightarrow \int_0^1 \int_{\sqrt[3]{x}}^1 e^{y^4}\,dy\,dx = \int_0^1 \int_0^{y^3} e^{y^4}\,dx\,dy = \int_0^1 x e^{y^4}\Big|_0^{y^3} dy$$

$$= \int_0^1 y^3 e^{y^4}\,dy = \int_0^1 \frac{e^u}{4}\,du = \frac{e^u}{4}\Big|_0^1 = \frac{e^{y^4}}{4}\Big|_0^1$$

$u = y^4$
$du = 4y^3 dy$

$$= \frac{e}{4} - \frac{1}{4} = \frac{e-1}{4}$$

⑨ Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ by reversing the order of integration.

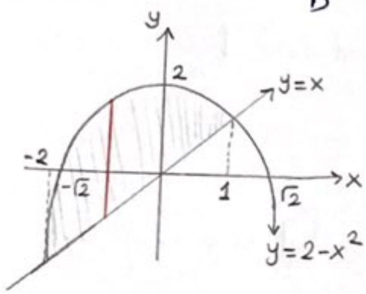
$$\left.\begin{array}{l} y = 0 \text{ to } 1 \\ x = 3y \text{ to } 3 \end{array}\right\} \Rightarrow \begin{array}{l} x = 0 \text{ to } 3 \\ y = 0 \text{ to } x/3 \end{array}$$

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 y e^{x^2}\Big|_0^{x/3} dx = \int_0^3 \frac{1}{3}x.e^{x^2}\,dx$$

$$= \frac{1}{6}e^{x^2}\Big|_0^3 = \frac{1}{6}e^9 - \frac{1}{6} = \frac{e^9-1}{6}$$

(10) Let D be a region bounded by $y=2-x^2$ and $y=x$.

a) Evaluate $\iint_D x^2\,dA$.



$x=2-x^2 \Rightarrow x^2+x-2=0 \Rightarrow x_1=1,\ x_2=-2$

Region (vertical) | $x=-2$ to 1
$y=x$ to $2-x^2$

$$A=\iint_D x^2\,dy\,dx=\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx=\int_{-2}^{1} x^2y\Big|_{x}^{2-x^2}dx=\int_{-2}^{1} x^2(2-x^2-x)\,dx$$

$$=\int_{-2}^{1}(2x^2-x^4-x^3)\,dx=\frac{2x^3}{3}-\frac{x^5}{5}-\frac{x^4}{4}\Big|_{-2}^{1}$$

$$=\left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{32}{5}-\frac{16}{4}\right)=\frac{63}{20}$$

b) Find the average value of $f(x,y)=x^2$ over the region D.

Area of D: $A=\int_{-2}^{1}\int_{x}^{2-x^2} dy\,dx=\int_{-2}^{1}(2-x^2-x)\,dx=2x-\frac{x^3}{3}-\frac{x^2}{2}\Big|_{-2}^{1}=\frac{9}{2}$

Average value of f over D $=\dfrac{1}{\text{Area of } D}\cdot\iint_D f\,dA$

$$=\frac{63/20}{9/2}=\frac{63}{20}\cdot\frac{2}{9}=\frac{7}{10}$$