Değişken katsayılı Lineer Dif. Denklemler

Euler - Cauchy Dif. Denklem

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + \dots + a_{n-2} x^2 y'' + a_{n-1} x y' + a_n y = f(x)$$

$i = 0, 1, 2, \dots, n \qquad a_i \in \mathbb{R}$

$\underbrace{x = e^t}$ değişken dönüşümü

$y = y(x) \qquad x = x(t)$

$$y = y(x(t)) = y(t)$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{dy}{dt}}{e^t}$$

$$\boxed{\frac{dy}{dx} = e^{-t}\frac{dy}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\underbrace{\frac{dy}{dx}}_{y'}\right) = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)}{e^t}$$

$$= \frac{-e^{-t}\frac{dy}{dt} + \frac{d^2y}{dt^2}e^{-t}}{e^t}$$

$$\boxed{\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\underbrace{\frac{d^2y}{dx^2}}_{y''}\right) = \frac{dy''}{dx} = \frac{\frac{dy''}{dt}}{\frac{dx}{dt}} = \dots$$

$$\boxed{\frac{d^3y}{dx^3} = e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)}$$

$x=e^t$

$$\cdots + a_{n-3}x^3y''' + a_{n-2}x^2y'' + a_{n-1}xy' + a_n y = f(x)$$

$$\cdots + a_{n-3}\underbrace{e^{3t}e^{-3t}}_{e^0=1}\left(\underbrace{\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2}} + 2\frac{dy}{dt}\right) + a_{n-2}\underbrace{e^{2t}e^{-2t}}_{e^0=1}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_{n-1}\underbrace{e^{t}e^{-t}}_{e^0=1}\frac{dy}{dt} + a_n y = f(e^t)$$

Değişken kat. → Sabit katsayılı Lineer dif. → Belirsiz kat. / Sabitlerin Değişimi

$$\frac{dy}{dt} = \underbrace{\frac{d}{dt}}(y) = Dy \qquad \frac{d^2y}{dt^2} = \underbrace{\frac{d}{dt}}_{D}\underbrace{\left(\frac{dy}{dt}\right)}_{Dy} = D^2y$$

$D = \frac{d}{dt}$ → Türev operatörü

$$\frac{d^3y}{dt^3} = D^3y \qquad D = \frac{d}{dt}$$

$x=e^t \quad y=y(t) \quad D=\frac{d}{dt}$

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt} \to \frac{dy}{dx} = e^{-t}Dy$$

$$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \to \frac{d^2y}{dx^2} = e^{-2t}(D^2y - Dy)$$

$$= e^{-2t}(D^2-D)y = e^{-2t}D(D-1)y$$

$$\frac{d^3y}{dx^3} = e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \to \frac{d^3y}{dx^3} = e^{-3t}D(D-1)(D-2)y$$

$x=e^t \quad y=y(x(t))$

$$\frac{dy}{dx} = e^{-t}Dy$$

$$\frac{d^2y}{dx^2} = e^{-2t}D(D-1)y$$

$$\frac{d^3y}{dx^3} = e^{-3t}D(D-1)(D-2)y$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = e^{-nt}D(D-1)(D-2)\cdots(D-n+1)y$$

Örnek $x^2y'' + 4xy' + 2y = \ln x$ Genel çözümü?

$\boxed{x = e^t}$ $D = \frac{d}{dt}$

$y' = e^{-t} D y$

$y'' = e^{-2t} D(D-1) y$

$e^{2t} e^{-2t} D(D-1)y + 4 e^{t} e^{-t} D y + 2y = \ln e^t$

$(D^2 - D + 4D + 2)y = t$

$(D^2 + 3D + 2)y = t$

$\boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t}$ sabit katsayılı lineer dif. denk.

$x = e^t$

$y' = e^{-t} \frac{dy}{dt}$

$y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

$e^{2t} e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 4 e^{t} e^{-t} \frac{dy}{dt} + 2y = \ln e^t$

$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 4\frac{dy}{dt} + 2y = t$

$\boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t}$ //*

$r^2 + 3r + 2 = 0$ $r_1 = -2$, $r_2 = -1$ $y_h = c_1 e^{-2t} + c_2 e^{-t}$

$y_p = at + b$

$y_p' = a$

$y_p'' = 0$

$0 + 3a + 2at + 2b = t$

$2a = 1$

$3a + 2b = 0$

$a = \frac{1}{2}$

$b = \frac{-3}{4}$

$y_p = \frac{t}{2} - \frac{3}{4}$

$y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-t} + \frac{t}{2} - \frac{3}{4}$

$x = e^t \Leftrightarrow \ln x = t \Rightarrow$ $\boxed{y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{\ln x}{2} - \frac{3}{4}}$

örnek* $x^2y'' + 2xy' - 12y = 0$; $\begin{Bmatrix} y(1) = 1 \\ y'(1) = 12 \end{Bmatrix}$

Euler-Cauchy

$x = e^t \quad D = \frac{d}{dt}$

$y' = e^{-t} Dy$

$y'' = e^{-2t} D(D-1) y$

$$\underbrace{e^{2t} e^{-2t}} D(D-1)y + 2\underbrace{e^{t} e^{-t}} Dy - 12y = 0$$

$$(D^2 - D + 2D - 12)y = 0$$

$$(D^2 + D - 12)y = 0$$

$$* \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 12y = 0$$

$r^2 + r - 12 = 0 \quad r_1 = 3 \quad r_2 = -4$

$$\boxed{y = C_1 e^{3t} + C_2 e^{-4t}}$$

$x = e^t \Leftrightarrow t = \ln x$

$$y = C_1 x^3 + \frac{C_2}{x^4} \; * \quad \text{Genel Çözüm}$$

$y(1) = 1 \Leftrightarrow x = 1; \; y = 1, \; y' = 12$

$y'(1) = 12$

$$\left.\begin{aligned} y &= C_1 x^3 + \frac{C_2}{x^4} \\ y' &= 3C_1 x^2 - \frac{4C_2}{x^5} \end{aligned}\right\} \quad \begin{aligned} C_1 + C_2 &= 1 \\ 3C_1 - 4C_2 &= 12 \end{aligned}$$

$C_1 = \frac{16}{7}$

$C_2 = -\frac{9}{7}$

$$\boxed{y = \frac{16}{7}x^3 - \frac{9}{7x^4}} \quad x \neq 0$$

örnek $x\left(x^2 y''' - 2xy'' = 2x^2 - 1\right)$

$$x^3 y''' - 2x^2 y'' = 2x^3 - x \; *$$

$x = e^t$

$y' = e^{-t} Dy$

$y'' = e^{-2t} D(D-1)y$

$y''' = e^{-3t} D(D-1)(D-2)y$

$-2D(D-1)y$

$$\underbrace{e^{3t} e^{-3t}} (D^3 - 3D^2 + 2D) - 2\underbrace{e^{2t} e^{-2t}} D(D-1)y = 2e^{3t} - e^t$$

$$(D^3 - 3D^2 + 2D - 2D^2 + 2D)y = 2e^{3t} - e^t$$

$$(D^3 - 5D^2 + 4D)y = 2e^{3t} - e^t$$

$$\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = \underbrace{2e^{3t}}_{1} - \underbrace{e^{t}}_{2}$$

$r^3 - 5r^2 + 4r = 0$

$r(r^2 - 5r + 4) = 0$

$r_1 = 0$, $r_2 = 1$, $r_3 = 4$ $\Big\}$ $y_h = C_1 + C_2 e^{t} + C_3 e^{4t}$

$y_{p_1} = A e^{3t}$

$A = -\frac{1}{3}$

$y_{p_1} = -\frac{1}{3}e^{3t}$

$y_{p_2} = B\, e^{t} \cdot t^1$ ← Rezonans var.

$B = \frac{1}{3}$ $\quad y_{p_2} = \frac{1}{3} t e^{t}$

$y_p = y_{p_1} + y_{p_2} \Rightarrow y = y_h + y_p$

$$y = C_1 + C_2 e^{t} + C_3 e^{4t} - \frac{1}{3}e^{3t} + \frac{t e^{t}}{3}$$

$x = e^{t} \Leftrightarrow t = \ln x$

$$y = c_1 + C_2 x + C_3 x^4 - \frac{x^3}{3} + \frac{x \ln x}{3}$$

Genel Çözüm!

SORU $\quad x^3 y'' + 4x^2 y' = -1$

Cevap: $y = C_1 + \frac{C_2}{x^3} + \frac{1}{2x}$

SORU $\quad x^2 y'' - 5xy' + 9y = \frac{x^3}{(\ln x)^2}$

Cevap:

$$y = x^3\left[\left(-\frac{1}{\ln x} + K_1\right)\ln x - \ln(\ln x) + K_2\right]$$