Fredholm'un iki temel bağıntısı

$$D(x,t;\lambda) - K(x,t)D(\lambda) = \lambda\int_a^b D(x,y;\lambda)K(y,t)\,dy \quad \text{1. bağ.}$$

$$D(x,t;\lambda) - K(x,t)D(\lambda) = \lambda\int_a^b K(x,y)D(y,t;\lambda)\,dy \quad \text{2. bağ.}$$

A_n ve $B_n(x,t)$

$$D(x,t;\lambda) = K(x,t) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lambda^n B_n(x,t)$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lambda^n A_n$$

Bunları 1. bağıntıda yerine koyalım

$$K(x,t) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lambda^n B_n(x,t) - K(x,t)\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lambda^n A_n\right]$$

$$= \lambda\int_a^b \left(K(x,y) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lambda^n B_n(x,y)\right)K(y,t)\,dy$$

$$\cancel{K(x,t)} - \lambda B_1(x,t) + \frac{\lambda^2}{2!}B_2(x,t) - \frac{\lambda^3}{3!}B_3(x,t) + \dots - \cancel{K(x,t)} + \lambda A_1K(x,t)$$

$$-\frac{\lambda^2}{2!}A_2K(x,t) + \frac{\lambda^3}{3!}A_3K(x,t) - \dots$$

$$= \lambda\int_a^b K(x,y)K(y,t)\,dy + \lambda\int_a^b\left\{-\lambda B_1(x,y)K(y,t) + \frac{\lambda^2}{2!}B_2(x,y)K(y,t) - \dots\right\}dy$$

$$\lambda\left[A_1K(x,t) - B_1(x,t)\right] - \frac{\lambda^2}{2!}\left[A_2K(x,t) - B_2(x,t)\right] + \frac{\lambda^3}{3!}\left[A_3K(x,t) - B_3(x,t)\right]$$

$$= \lambda\int_a^b K(x,y)K(y,t)\,dy - \lambda^2\int_a^b B_1(x,y)K(y,t)\,dy + \frac{\lambda^3}{2!}\int_a^b B_2(x,y)K(y,t)\,dy - \dots$$

$$\int_a^b \overbrace{K(x,y)}^{B_0(x,t)} K(y,t)\,dy = A_1 K(x,t) - B_1(x,t)$$

$$\int_a^b B_1(x,y) K(y,t)\,dy = \frac{1}{2!}\left[A_2 K(x,t) - B_2(x,t)\right]$$

$$\frac{1}{2!}\int_a^b B_2(x,y) K(y,t)\,dy = \frac{1}{3!}\left[A_3 K(x,t) - B_3(x,t)\right]$$

$$\vdots$$

$$\frac{1}{(n-1)!}\int_a^b B_{n-1}(x,y) K(y,t)\,dy = \frac{1}{n!}\left[A_n K(x,t) - B_n(x,t)\right]$$

$$\boxed{B_n(x,t) = A_n K(x,t) - n\int_a^b B_{n-1}(x,y) K(y,t)\,dy}$$

Aynı işlemler 2. bağıntıdan hareket ederek tekrar edilirse

$$B_n(x,t) = A_n K(x,t) - n\int_a^b K(x,y) B_{n-1}(y,t)\,dy$$

bulunur.

$$B_n(x,t) = \int_a^b \cdots (n) \cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & \vdots & & \vdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

Rekürens Bağıntıları

$$B_n(x,t) = A_n K(x,t) - n\int_a^b B_{n-1}(x,y) K(y,t)\,dy \qquad (1)$$

$$B_{n-1}(x,t)=\int_a^b\cdots(n-1)\cdots\int_a^b\begin{vmatrix}K(x,t) & K(x,t_1) & \cdots & K(x,t_{n-1})\\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_{n-1})\\ \vdots & \vdots & & \vdots\\ K(t_{n-1},t) & K(t_{n-1},t_1) & \cdots & K(t_{n-1},t_{n-1})\end{vmatrix}dt_1\cdots dt_{n-1}$$

$t=x$ alınırsa

$$B_{n-1}(x,x)=\int_a^b\cdots(n-1)\cdots\int_a^b\begin{vmatrix}K(x,x) & K(x,t_1) & \cdots & K(x,t_{n-1})\\ K(t_1,x) & K(t_1,t_1) & \cdots & K(t_1,t_{n-1})\\ \vdots & \vdots & & \vdots\\ K(t_{n-1},x) & K(t_{n-1},t_1) & \cdots & K(t_{n-1},t_{n-1})\end{vmatrix}=dt_1\cdots dt_{n-1}$$

olup determinantın 1. satırı n. satıra, 1. sütunu n. sütuna götürülürse det. işareti değişmeyecektir. $x=t_n$ almak suretiyle $dx=dt_n$ olacağından $B_{n-1}(x,x)$ integre edilirse

$$\int_a^b B_{n-1}(x,x)dx=\underbrace{\int_a^b\int_a^b\cdots(n-1)\cdots\int_a^b\begin{vmatrix}K(t_1,t_1) & \cdots & K(t_1,t_n)\\ \vdots & & \vdots\\ K(t_n,t_1) & \cdots & K(t_n,t_n)\end{vmatrix}dt_1\cdots dt_{n-1}dt_n}$$

$$\boxed{A_n=\int_a^b B_{n-1}(x,x)\,dx} \quad (2)$$

Özel olarak $n=1$ için $A_1=\int_a^b B_0(x,x)dx=\int_a^b K(x,x)dx$ olur.

$B_0(x,t)=K(x,t)$ old. biliyoruz. $n=0$ alırsak

$$B_0(x,t)=\cancel{K(x,t)}=A_0\cancel{K(x,t)}$$

$$A_0=1$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n A_n$$

$$D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!} A_2 - \frac{\lambda^3}{3!} A_3 + \cdots + (-1)^n \frac{\lambda^n}{n!} A_n + \cdots$$

$$\frac{dD(\lambda)}{d\lambda} = -A_1 + \lambda A_2 - \frac{\lambda^2}{2!} A_3 + \cdots + (-1)^n \frac{\lambda^{n-1}}{(n-1)!} A_n + \cdots$$

$$\frac{dD(\lambda)}{d\lambda} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n!} A_{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n!} \int_a^b B_n(x,x)\, dx$$

$$= \int_a^b \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n!} B_n(x,x)\, dx$$

$$= \int_a^b \left[-B_0(x,x) - \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} B_n(x,x) \right] dx$$

$$= -\int_a^b \left[K(x,x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n B_n(x,x) \right] dx$$

$$\boxed{D'(\lambda) = -\int_a^b D(x,x;\lambda)\, dx} \quad (3)$$

$$\frac{D'(\lambda)}{D(\lambda)} = -\frac{1}{D(\lambda)}\int_a^b D(x,x;\lambda)\,dx = -\int \underbrace{\frac{D(x,x,\lambda)}{D(\lambda)}}\,dx$$

$$\boxed{\frac{D'(\lambda)}{D(\lambda)} = -\int \Gamma(x,x;\lambda)\,dx}$$

$$\frac{D'(\lambda)}{D(\lambda)} = -\int_a^b \left\{K(x,x)+\lambda K_{(2)}(x,x)+\lambda^2 K_{(3)}(x,x) \quad +\dots+\lambda^{n-1} K_{(n)}(x,x)+\dots\right\}dx$$

$$= -\underbrace{\int_a^b K(x,x)\,dx}_{I_1} - \lambda\underbrace{\int_a^b K_{(2)}(x,x)\,dx}_{I_2} - \lambda^2 \underbrace{\int_a^b K_{(3)}(x,x)\,dx}_{I_3} - \dots - \lambda^{n-1}\underbrace{\int_a^b K_{(n)}(x,x)\,dx}_{I_n} - \dots$$

Bunlara $K(x,t)$'nin izleri denir. I_n'e $K(x,t)$'nin n. izi denir.

$$\frac{D'(\lambda)}{D(\lambda)} = -\left(I_1+\lambda I_2+\lambda^2 I_3+\dots+\lambda^{n-1} I_n+\dots\right)$$

$$\frac{d}{d\lambda}\left[\ln D(\lambda)\right] = -\left(I_1+\lambda I_2+\lambda^2 I_3+\dots+\lambda^{n-1} I_n+\dots\right)$$

$$\int d\left[\ln(D(\lambda))\right] = -\int\left(\qquad '' \qquad \right)d\lambda$$

$$\ln D(\lambda) = -I_1\lambda - I_2\frac{\lambda^2}{2} - I_3\frac{\lambda^3}{3} - \dots$$

$$D(\lambda) = e^{-\left(I_1\lambda + I_2\frac{\lambda^2}{2} + I_3\frac{\lambda^3}{3}+\dots\right)}$$

Karşıt (Reciprocal) Fonksiyon

$K(x,t)$ çekirdek fonk.nunun itere çekirdekleri

$K_{(1)}(x,t), K_{(2)}(x,t), \ldots$ olmak üzere

$$-k(x,t) = K_{(1)}(x,t) + K_{(2)}(x,t) + \ldots = \sum_{n=1}^{\infty} K_{(n)}(x,t)$$

şeklinde tanımlanan fonk. karşıt fonksiyon denir.

$$-k(x,t) = \underbrace{K_{(1)}(x,t)}_{K(x,t)} + K_{(2)}(x,t) + \cdots + K_{(n)}(x,t) + \ldots$$

$$K_{(n)}(x,t) = \int_a^b K(x,y) K_{(n-1)}(y,t)\,dy$$

$$-k(x,t) - K(x,t) = \int_a^b K_{(1)}(x,y) K_{(1)}(y,t)\,dy + \int_a^b K_{(1)}(x,y) K_{(2)}(y,t)\,dy \ldots$$

$$+ \int_a^b K_{(1)}(x,y) K_{(n-1)}(y,t)\,dy + \ldots$$

$$-k(x,t) - K(x,t) = \int_a^b K_{(1)}(x,y) \Big[\underbrace{K_{(1)}(y,t) + K_{(2)}(y,t) + \cdots + K_{(n)}(y,t)}_{-k(y,t)} \Big] dy$$

$$\boxed{k(x,t) + K(x,t) = \int_a^b K(x,y)\, k(y,t)\,dy}$$

Örnek $K(x,t)=x^2t^2$ $\quad a=0$, $b=1$ Karşıt fonk.?

$$K_{(1)}(x,t)=x^2t^2$$

$$K_{(2)}(x,t)=\int_0^1 K(x,y)\,K(y,t)\,dy=\int_0^1 x^2y^2y^2t^2\,dy=x^2t^2\int_0^1 y^4\,dy$$

$$=\frac{1}{5}x^2t^2$$

$$K_{(3)}(x,t)=\int_0^1 K(x,y)\,K_{(2)}(y,t)\,dy=\frac{1}{5}\int_0^1 x^2y^2y^2t^2\,dy=\frac{1}{5^2}x^2t^2$$

$\vdots$

$$K_{(n)}(x,t)=\frac{1}{5^{n-1}}x^2t^2$$

$$-k(x,t)=K_{(1)}(x,t)+K_{(2)}(x,t)+\dots+K_{(n)}(x,t)+\dots$$

$$=x^2t^2+\frac{1}{5}x^2t^2+\frac{1}{5^2}x^2t^2+\dots$$

$$=x^2t^2\underbrace{\left(1+\frac{1}{5}+\frac{1}{5^2}+\dots\right)}_{\frac{5}{4}}$$ $r=\frac{1}{5}<1$ yakınsak

$$\boxed{k(x,t)=-\frac{5}{4}x^2t^2}$$