Volterra Yöntemi

$$u(x) = f(x) + \int_a^b K(x,t)\,u(t)\,dt \quad (1)$$

$$u(t) = f(t) + \int_a^b K(t,y)\,u(y)\,dy$$

$K(x,t)$ nin karşıt fonk.nu $k(x,t)$ olmak üzere

$$\int_a^b k(x,t)\,u(t)\,dt = \int_a^b k(x,t) f(t)\,dt + \int_a^b \int_a^b \underbrace{k(x,t) K(t,y)}\,u(y)\,dy\,dt$$

$$\int_a^b k(x,t) u(t)\,dt = \int_a^b k(x,t) f(t)\,dt + \int_a^b \left[k(x,y) + K(x,y)\right] u(y)\,dy$$

$$\cancel{\int_a^b k(x,t) u(t)\,dt} = \int_a^b k(x,t) f(t)\,dt + \cancel{\int_a^b k(x,y) u(y)\,dy} + \int_a^b K(x,y) u(y)\,dy$$

$$\int_a^b k(x,t) f(t)\,dt + \int_a^b K(x,t) u(t)\,dt = 0$$

$$\int_a^b K(x,t) u(t)\,dt = -\int_a^b k(x,t) f(t)\,dt$$

(1) de yerine koyarsak

$$u(x) = f(x) - \int_a^b k(x,t) f(t)\,dt$$

Volterra yönt. ancak $\lambda=1$ için denklemin çözümü söz konusu olacağından

$$\Gamma(x,t;\lambda)=\Gamma(x,t;1)$$

$$\Gamma(x,t;1)=\sum_{n=1}^{\infty}K_{(n)}(x,t)=-k(x,t)$$

$$u(x)=f(x)+\int_a^b \Gamma(x,t;1)f(t)\,dt$$

$$u(x)=f(x)+\int_a^b\left[-k(x,t)\right]f(t)\,dt$$

Örnek

$$u(x)=x+\int_0^1 xt\,u(t)\,dt \quad \text{Volterra yönt.}$$

$$K_{(1)}(x,t)=xt$$

$$K_{(2)}(x,t)=\int_0^1 K(x,y)K_{(1)}(y,t)\,dy=\int_0^1 xy\,yt\,dy=\frac{xt}{3}$$

$$K_{(3)}(x,t)=\int_0^1 K(x,y)K_{(2)}(y,t)\,dy=\int_0^1 xy\frac{yt}{3}\,dy=\frac{xt}{3^2}$$

$\vdots$

$$K_{(n)}(x,t)=\frac{xt}{3^{n-1}}$$

$$-\ k(x,t)=\sum_{n=1}^{\infty}K_{(n)}(x,t)$$

$$u(x)=f(x)-\int_a^b k(x,t)f(t)\,dt$$

$$=xt\left[1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right]$$

$r=\frac{1}{3}<1$ yakınsak

$$=xt\ \frac{1}{1-\frac{1}{3}}=\frac{3xt}{2}$$

$$u(x)=x+\int_0^1 xt\,u(t)\,dt=x+\int_0^1\frac{3}{2}xt\ \underset{f(t)}{t}\,dt$$

$$=\frac{3x}{2}\ ✓$$

Volterra İntegral Denklemi

$u(x)$ bilinmeyen fonk. $K(x,t)$ çekirdek fonk

λ parametre

$$\phi(x) = \lambda \int_a^x K(x,t)u(t)\,dt \qquad \text{1. cins Volterra}$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt \qquad \text{2. cins Volterra}$$

$$f(x) \equiv 0$$

$$u(x) = \lambda \int_a^x K(x,t)u(t)\,dt \qquad \text{2. cins homojen Volterra}$$

Dif. Denkleme dönüştürerek Çözümü:

$$f(x) = \int_{a(x)}^{b(x)} F(x,t)\,dt$$

$$f'(x) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[F(x,t)\right]dt + \frac{db}{dx}\left(F(x,b(x))\right) - \frac{da}{dx} \cdot F(x,a(x))$$

Örnek

$$2x^3 = \int_0^x \left(\left[1 - 5(x-t) + 2(x-t)^2\right] . u(t)\right)dt$$

* $$6x^2 = \int_0^x \left(-5 + 4(x-t)\right)u(t)\,dt + u(x)$$

$$* \quad 12x = \int_{0}^{x} 4u(t)\,dt - 5u(x) + u'(x)$$

$$12 = 4u(x) - 5u'(x) + u''(x)$$

$$u''(x) - 5u'(x) + 4u(x) = 12$$

2. mertebeden sabit katsayılı lineer dif denk.

$$\left.\begin{aligned} u(0) &= 0 \\ u'(0) &= 0 \end{aligned}\right\} \text{bulmacak!}$$

$$u(x) = c_1 e^{x} + c_2 e^{4x} + 3$$

$$u'(x) = c_1 e^{x} + 4c_2 e^{4x}$$

$$\left.\begin{aligned} x = 0 \quad u &= 0 \\ u' &= 0 \end{aligned}\right\}$$

$$c_2 = 1 \qquad c_1 = -4$$

$$\boxed{u(x) = -4e^{x} + e^{4x} + 3}$$

$$c_1 + c_2 + 3 = 0$$

$$\cancel{c_1} + c_2 = -3$$

$$- \cancel{c_1} \mp 4c_2 = 0$$

$$-3c_2 = -3$$

Çözücü Çekirdek Yardımıyla Çözüm:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt$$ 2. cins Volterra

Bu denklemin $u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \dots$ (2)

şeklinde çözümü araştırılacaktır.

$$u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \dots = f(x) + \lambda \int_0^x K(x,t) \left[u_0(t) + \lambda u_1(t) + \lambda^2 u_2(t) + \dots \right] dt$$

$$u_0(x) = f(x)$$

$$u_1(x) = \int_0^x K(x,t) \underbrace{u_0(t)}_{f(t)}\, dt$$

$$u_2(x) = \int_0^x K(x,t)\, u_1(t)\, dt$$

$$\vdots$$

Böylece
$$\begin{cases} u_0(x) = f(x) \\ u_n(x) = \int_0^x K(x,t)\, u_{n-1}(t)\, dt & (n \geq 1) \end{cases}$$

$u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x)$ serinin yakınsaklığı ile sonuca ulaşılır.

İtere Çekirdek

$K_{(1)}(x,t) = K(x,t)$ olmak üzere

$$K_{(n)}(x,t) = \int_t^x K(x,y)\,K_{(n-1)}(y,t)\,dy$$

$K_{(n)}(x,t)$ $\quad u_2(x)$

$$u_n(x) = \int_0^x K_{(n)}(x,t)\,f(t)\,dt$$

Buradan ② denklemini yeniden düzenlersek

$$u(x) = f(x) + \sum_{i=1}^{\infty} \lambda^i \int_0^x K_{(i)}(x,t)\,f(t)\,dt \quad \text{bulunur.}$$

$K(x,t)$ sürekli fonk olduğundan

$$\sum_{i=1}^{\infty} \lambda^{i-1} K_{(i)}(x,t) \quad \text{mutlak ve düzgün yak. olup}$$

Resolvant denir.

$$R(x,t;\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{(i)}(x,t)$$

$$u(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)\,f(t)\,dt$$

ile çözüme ulaşılır.

Örnek

$$u(x) = x + \int_0^x u(t)\,dt$$

$$K_{(1)}(x,t) = K(x,t) = 1$$

$$K_{(2)}(x,t) = \int_t^x K(x,y)K_{(1)}(y,t)\,dy = \int_t^x 1.1\,dy = x-t$$

$$K_{(3)}(x,t) = \int_t^x K(x,y)K_{(2)}(y,t)\,dy = \int_t^x 1.(y-t)\,dy = \frac{(x-t)^2}{2}$$

$$K_{(4)}(x,t) = \int_t^x K(x,y)K_{(3)}(y,t)\,dt = \int_t^x 1.\frac{(y-t)^2}{2}\,dy = \frac{(x-t)^3}{3.2}$$

$$\vdots \qquad\qquad \vdots$$

$$K_{(n)}(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}$$

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\frac{(x-t)^{n-1}}{(n-1)!}$$

$$= 1 + \lambda(x-t) + \frac{(\lambda(x-t))^2}{2!} + \cdots$$

$$= e^{\lambda(x-t)}$$

$$R(x,t;1) = e^{x-t}$$

$$u(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)\, dt$$

$$u(x) = x + \int_0^x e^{x-t} \cdot t\, dt = x + e^x \int_0^x t e^{-t}\, dt$$

$$\boxed{u(x) = e^x - 1}$$

$$\left.\begin{aligned} e^u &= 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \\ \sin u &= u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots \\ \cos u &= 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots \end{aligned}\right\} \forall u \in \mathbb{R}$$

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots \qquad |u| < 1$$

SORU $K(x,t) = x - t$ Volterra int. denklemine ait rezolvent?

$$R(x,t;\lambda) = \frac{1}{\sqrt{\lambda}} \sinh\left[\sqrt{\lambda}(x-t)\right]$$

$$\sinh(u) = \frac{e^u - e^{-u}}{2} \qquad \cosh(u) = \frac{e^u + e^{-u}}{2}$$