SABİT KATSAYILI DİF. DENKLEMLERİN LAPLACE DÖNÜŞÜMÜ İLE ÇÖZÜMÜ

Laplace Dönüşümleri:

Tanım: $K(s,t) = \begin{cases} 0 & : t<0 \\ e^{-st} & : t \geq 0 \end{cases}$ çekirdek fonk.

$\boxed{f(t)}$ verilsin.

$$\int_{-\infty}^{\infty} K(s,t).f(t)\,dt = \int_{-\infty}^{0} \underbrace{K(s,t)}_{0} f(t)\,dt + \int_{0}^{\infty} \underbrace{K(s,t)}_{e^{-st}} f(t)\,dt$$

$$= \int_{0}^{\infty} e^{-st} f(t)\,dt = F(s)$$

$$L\{f(t)\} = \int_{0}^{\infty} e^{-st}.f(t)\,dt = F(s)$$

$$\boxed{L\{f(t)\} = F(s)}$$

Bu ifadeye $f(t)$ fonk.nun Laplace dönüşümü denir.

$t \longrightarrow s$

Bazı elemanter fonksiyonların Laplace dönüşümlerini verelim:

$$\int e^{-2t}\,dt = -\frac{1}{2} e^{-2t} + c$$

1) $f(t) = c \quad (c \in \mathbb{R})$

$$L\{c\} = \int_0^{\infty} c.e^{-st}\,dt = c\int_0^{\infty} e^{-st}\,dt = c.\lim_{A\to\infty} \int_0^{A} e^{-st}\,dt$$

$$= c.\lim_{A\to\infty}\left[-\frac{1}{s} e^{-st}\right]_0^A = -\frac{c}{s}.\lim_{A\to\infty}\left(\underbrace{e^{-sA}}_{0} - 1\right) \quad (s>0)$$

$= \frac{c}{s}$ $\boxed{L\{c\} = \frac{c}{s}}$

$L\{-3\} = \frac{-3}{s}$ $L\{\sqrt{2}\} = \frac{\sqrt{2}}{s}$ gibi...

2) $f(t) = t \quad (s>0)$ $\int t e^{-st} dt$ kısmi int.

$$L\{t\} = \int_0^\infty t e^{-st} dt = \lim_{A\to\infty} \int_0^A t e^{-st} dt = \frac{1}{s^2} = \frac{1!}{s^2}$$

$$L\{t^2\} = \int_0^\infty t^2 e^{-st} dt = \frac{1\cdot 2}{s^3} = \frac{2!}{s^3}$$

$$L\{t^3\} = \underbrace{\int_0^\infty t^3 e^{-st} dt}_{f(t)} = \frac{1\cdot 2\cdot 3}{s^4} = \frac{3!}{s^4}$$

$\vdots$

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}} \quad (s>0)$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

3) $f(t) = e^{at} \quad \underbrace{(s>a)}_{s-a>0} \quad (a\in\mathbb{R})$

$$L\{e^{at}\} = \int_0^\infty e^{at}\cdot e^{-st} dt = \lim_{A\to\infty} \int_0^A e^{\underbrace{-(s-a)t}} dt$$

$$= \frac{-1}{s-a} \lim_{A\to\infty} \left[e^{-(s-a)t} \right]_0^A = \frac{-1}{s-a} \lim_{A\to\infty} \left(\underbrace{e^{-(s-a)A}}_{0} - 1 \right) = \frac{1}{s-a}$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$L\{t^6\} = \frac{6!}{s^7} \qquad L\{e^{3t}\} = \frac{1}{s-3}$$

$$L\{t^{10}\} = \frac{10!}{s^{11}} \qquad L\{e^{-3t}\} = \frac{1}{s+3}$$

4) $f(t) = \sin at \quad (a \in \mathbb{R})$

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\, dt = \frac{a}{s^2+a^2}$$

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt = \frac{s}{s^2+a^2}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}} \qquad \boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

$$L\{\sin 4t\} = \frac{4}{s^2+16} \qquad L\{\cos 4t\} = \frac{s}{s^2+16}$$

5) $f(t) = \sinh(at) \quad a \in \mathbb{R} \qquad \sinh at = \frac{e^{at} - e^{-at}}{2}$

$$\boxed{L\{\sinh(at)\} = \frac{a}{s^2-a^2}}$$

$f(t) = \cosh(at) \quad a \in \mathbb{R} \qquad \cosh(at) = \frac{e^{at} + e^{-at}}{2}$

$$\boxed{L\{\cosh(at)\} = \frac{s}{s^2-a^2}} \qquad L\{\cosh 2t\} = \frac{s}{s^2-4}$$

$$L\{\sinh t\} = \frac{1}{s^2-1}$$

Laplace Dönüşümünün Özellikleri

1) Lineerlik özelliği: $c_1, c_2, c_3, \dots \in \mathbb{R}$ olmak üzere

$$L\{c_1 f_1(t) \mp c_2 f_2(t) \mp c_3 f_3(t) \mp \dots\} = c_1 L\{f_1(t)\} \mp c_2 L\{f_2(t)\} \mp \dots$$

Örnek $L\{4t^3 - 3\sin 5t - 7e^{-4t}\}$

$$= 4L\{t^3\} - 3L\{\sin 5t\} - 7L\{e^{-4t}\}$$

$$= 4\,\frac{3!}{s^4} - 3\,\frac{5}{s^2+25} - 7\cdot\frac{1}{s+4}$$

2) Kaydırma özelliği:

$L\{f(t)\} = F(s)$ ise $L\{e^{at} f(t)\} = F(s-a)$

Örnek; a) $L\{e^{-t}\cos 2t\} = \dfrac{s+1}{(s+1)^2+4}$

$L\{\cos 2t\} = \dfrac{s}{s^2+4}$

b) $L\{t^2 e^{3t}\} = \dfrac{2!}{(s-3)^2}$

$L\{t^2\} = \dfrac{2!}{s^3}$

3) Türevin Laplace Dönüşümü:

$L\{f(t)\} = F(s)$ ise

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - s f(0) - f'(0)$$

$$L\{f'''(t)\} = s^3F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\vdots$$

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

örnek $f(t)=\cos 3t$ $\quad L\{\cos 3t\}=\frac{s}{s^2+9}$ $\quad f(0)=1$

$$L\{f'(t)\}=sF(s)-f(0)$$
$$=s\frac{s}{s^2+9}-1=\frac{s^2}{s^2+9}-1=\frac{s^2-s^2-9}{s^2+9}$$
$$=\frac{-9}{s^2+9}$$

$f(t)=\cos 3t$

$f'(t)=-3\sin 3t$ $\quad L\{-3\sin 3t\}=-3\frac{3}{s^2+9}=\frac{-9}{s^2+9}$

4) t^n ile çarpma

$L\{f(t)\}=F(s)$ ise $L\{t^n.f(t)\}=(-1)^n.\frac{d^nF(s)}{ds^n}$

örnek $L\{t^2.e^{3t}\}$ $\quad L\{e^{3t}\}=\frac{1}{s-3}$

$$L\{t^2e^{3t}\}=(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)=\frac{2}{(s-3)^3}$$

2.yol: $L\{e^{3t}t^2\}$ $\quad L\{t^2\}=\frac{2!}{s^3}$

$$L\{e^{3t}t^2\}=\frac{2!}{(s-3)^3}$$

örnek $L\{t^1\sin t\}=(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\cancel{-}\frac{\cancel{-}2s}{(s^2+1)^2}=\frac{s}{(s^2+1)^2}$

$L\{\sin t\}=\frac{1}{s^2+1}$

Örnek: $L\{(t+2)^2 e^t\} = ?$

$$L\{t^2\} = \frac{2!}{s^3}$$

$$L\{t\} = \frac{1}{s^2}$$

$$L\{(t+2)^2 e^t\} = L\{(t^2+4t+4)e^t\}$$

$$= L\{\underbrace{t^2e^t}+\underbrace{4te^t}+\underbrace{4e^t}\}$$

$$= L\{t^2e^t\} + 4L\{te^t\} + 4L\{e^t\}$$

$$= \frac{2}{(s-1)^3} + 4\frac{1}{(s-1)^2} + 4\frac{1}{s-1}$$

Örnek:

$$L\{e^t \sin^2 3t\} = L\left\{e^t\left(\frac{1-\cos 6t}{2}\right)\right\}$$

$$= L\left\{e^t\left(\frac{1}{2}-\frac{1}{2}\cos 6t\right)\right\}$$

$$L\{\cos 6t\} = \frac{s}{s^2+36}$$

$$= L\left\{\frac{1}{2}e^t - \frac{1}{2}e^t\cos 6t\right\}$$

$$= \frac{1}{2}L\{e^t\} - \frac{1}{2}L\{e^t\cos 6t\}$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{s-1}{(s-1)^2+36}$$