İtere Çekirdek

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$ Fredholm int. denk.

$K(x,t)$ çekirdek fonk. $\qquad K_{(n)}(x,t)$

$K_{(1)}(x,t)$, $K_{(2)}(x,t)$, $K_{(3)}(x,t)$, ...

1. itere çek $\quad$ 2. itere çek. $\quad$ 3. itere çek.

$$K_{(1)}(x,t) = K(x,t)$$

$$K_{(2)}(x,t) = \int_a^b K(x,y)K(y,t)dy = \int_a^b K(x,y)K_{(1)}(y,t)dy$$

$$K_{(3)}(x,t) = \int_a^b\int_a^b K(x,y)K(y,z)K(z,t)dydz$$

$$= \int_a^b K(x,y)\underbrace{\left[\int_a^b K(y,z)K(z,t)dz\right]}dy$$

$$= \int_a^b K(x,y)K_{(2)}(y,t)dy$$

$$K_{(4)}(x,t) = \int_a^b K(x,y)K_{(3)}(y,t)dy$$

$\vdots$

$$\boxed{K_{(n)}(x,t) = \int_a^b K(x,y).K_{(n-1)}(y,t)dy}$$

Örnek $K(x,t) = x - t$ $\left.\begin{matrix} a=0 \\ b=1 \end{matrix}\right\}$ itere çekirdekler?

$$K_{(1)}(x,t) = x - t$$

$$K_{(2)}(x,t) = \int_0^1 K(x,y) K_{(1)}(y,t)\,dy = \int_0^1 (x-y)(y-t)\,dy$$

$$= \int_0^1 (xy - xt - y^2 + yt)\,dy = \left(\frac{xy^2}{2} - xty - \frac{y^3}{3} + \frac{y^2}{2}t\right)\Big|_0^1$$

$$= \frac{x+t}{2} - xt - \frac{1}{3}$$

$$K_{(3)}(x,t) = \int_0^1 K(x,y) K_{(2)}(y,t)\,dy$$

$$= \int_0^1 (x-y)\left(\frac{y+t}{2} - yt - \frac{1}{3}\right)dy$$

$$= -\frac{(x-t)}{12}$$

$$K_{(4)}(x,t) = \int_0^1 K(x,t) K_{(3)}(y,t)\,dy = \frac{-1}{12}\int_0^1 (x-y)(y-t)\,dy$$

$$= -\frac{1}{12}\left(\frac{x+t}{2} - xt - \frac{1}{3}\right)$$

$2 \to 0$

$4 \to 1$

$6 \to 2$

$$K_{(5)}(x,t) = \int_0^1 K(x,y) K_{(4)}(y,t)\,dy$$

$$= -\frac{1}{12}\int_0^1 (x-y)\left(\frac{y+t}{2} - yt - \frac{1}{3}\right)dy = \left(\frac{-1}{12}\right)^2 (x-t)$$

$$K_{(6)}(x,t) = \left(\frac{-1}{12}\right)^2 \left(\frac{x+t}{2} - xt - \frac{1}{3}\right)$$

$$K_{(7)}(x,t) = \left(-\frac{1}{12}\right)^3 (x-t)$$

$$\vdots$$

n tek $\quad K_{(n)}(x,t) = \left(\frac{-1}{12}\right)^{\frac{n+1}{2}-1} (x-t)$

n çift $\quad K_{(n)}(x,t) = \left(\frac{-1}{12}\right)^{\frac{n}{2}-1} \left(\frac{x+t}{2} - xt - \frac{1}{3}\right)$

örnek $K(x,t) = e^{x-t} \qquad a=0, \quad b=0.1$

$$K_{(1)}(x,t) = e^{x-t}$$

$$K_{(2)}(x,t) = \int_0^{0.1} K(x,y)\,K(y,t)\,dy = \int_0^{0.1} e^{x-y} \cdot e^{y-t}\,dy$$

$$= \int_0^{0.1} e^{x-t}\,dy = 0.1\,e^{x-t}$$

$$K_{(3)}(x,t) = \int_0^{0.1} K(x,y)\,K_{(2)}(y,t)\,dy = \int_0^{0.1} e^{x-y}\,0.1\,e^{y-t}\,dy$$

$$= (0.1)^2 e^{x-t}$$

$$K_{(4)}(x,t) = (0.1)^3 e^{x-t}$$

$$\vdots$$

$$K_{(n)}(x,t) = (0.1)^{n-1} e^{x-t} \qquad (n=1,2,\dots)$$

Neumann Serisi

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

$$u_0(x) = f(x) \qquad \left(u_1(x) = f(x) + \lambda \int_a^b K(x,t)\, u_0(t)\, dt \right)$$

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt$$

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)\, u_1(t)\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t) \left[f(t) + \lambda \int_a^b K(t,t_1) f(t_1)\, dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b \underbrace{\int_a^b K(x,t) K(t,t_1)}_{K_{(2)}(x,t_1)} f(t_1)\, \cancel{dt_1}\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K_{(2)}(x,t) f(t)\, dt$$

$$u_3(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K_{(2)}(x,t) f(t)\, dt + \lambda^3 \int_a^b K_{(3)}(x,t) f(t)\, dt$$

$$\vdots$$

$$u_n(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \dots + \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt$$

$$u(x) = \lim_{n \to \infty} u_n(x)$$

$$u(x) = \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \left[f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \dots + \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt \right]$$

$$u(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_{(n)}(x,t) f(t)\,dt$$

Neumann Serisi

$$u(x) = f(x) + \lambda \int_a^b K_{(1)}(x,t) f(t)\,dt + \lambda^2 \int_a^b K_{(2)}(x,t) f(t)\,dt + \cdots + \lambda^n \int_a^b K_{(n)}(x,t) f(t)\,dt + \cdots$$

$$u(x) = f(x) + \lambda \left[\int_a^b K_{(1)}(x,t) f(t)\,dt + \lambda \int_a^b K_{(2)}(x,t) f(t)\,dt + \cdots + \lambda^{n-1} \int_a^b K_{(n)}(x,t) f(t)\,dt + \cdots \right]$$

$$u(x) = f(x) + \lambda \int_a^b \underbrace{\left[K_{(1)}(x,t) + \lambda K_{(2)}(x,t) + \lambda^2 K_{(3)}(x,t) + \cdots \right]}_{\Gamma(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_{(n)}(x,t)} f(t)\,dt$$

Neumann Serisinin Yakınsaklığı:

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t)\,dt$$

$f(x)$ $a \leq x \leq b$ de tanımlı ve sürekli fonk. olduğundan

$|f(x)| < N$ olacak şekilde $N \in \mathbb{R}$ vardır.

$K(x,t)$ $\begin{array}{l} a \leq x \leq b \\ a \leq t \leq b \end{array}$ tanımlı ve sürekli fonk.

$|K(x,t)| < M$ olacak şekilde $M \in \mathbb{R}$ vardır

$$\lambda \int_a^b \underbrace{K(x,t)}\,\underbrace{f(t)}\,dt$$

$$\left| \lambda \int_a^b K(x,t) f(t)\, dt \right| \leq |\lambda|\, M \cdot N \cdot |b-a|$$

$$K_{(2)}(x,t) = \int_a^b \underbrace{K(x,y)} \cdot \underbrace{K(y,t)}\, dy \qquad \left| K_{(2)}(x,t) \right| < M^2 |b-a|$$

$$\left| \lambda^2 \int_a^b K_{(2)}(x,t) f(t)\, dt \right| \leq |\lambda^2|\, M^2 N\, |b-a|^2$$

$$\vdots$$

$$\left| \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt \right| \leq |\lambda^n|\, M^n N\, |b-a|^n$$

$$u_n(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K_{(2)}(x,t) f(t)\, dt + \dots + \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt$$

$$u(x) = \lim_{n \to \infty} u_n = f(x) + \underbrace{\sum_{n=1}^{\infty} \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt}$$

$$\sum_{n=1}^{\infty} \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt \leq |\lambda|\, MN\, |b-a| + |\lambda^2|\, M^2 N\, |b-a|^2 + \dots + |\lambda^n|\, M^n N\, |b-a|^n + \dots$$

$$= N\left(M|\lambda(b-a)| + M^2 |\lambda^2 (b-a)^2| + \dots \right)$$

$$\underbrace{\sum_{n=1}^{\infty} \lambda^n \int_a^b K_{(n)}(x,t) f(t)\, dt} \leq N \underbrace{\sum_{n=1}^{\infty} \left(M |\lambda(b-a)| \right)^n}_{\text{Geometrik}}$$

$$M\,|\lambda(b-a)| < 1$$

Yakınsak şartı

$$|\lambda| < \frac{1}{M|b-a|}$$

örnek

$$u(x) = e^{-x} + \lambda \int_0^{10} e^{x+t} u(t)\,dt$$ Neumann serisi ile çözünüz

$$\Gamma(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_{(n)}(x,t)$$

$$K_{(1)}(x,t) = e^{x+t}$$

$$K_{(2)}(x,t) = \int_0^{10} K(x,y) K_{(1)}(y,t)\,dy = \int_0^{10} e^{x+y} e^{y+t}\,dy$$

$$= e^{x+t}\int_0^{10} e^{2y}\,dy = e^{x+t}\left(\frac{1}{2}e^{2y}\right)\Big|_0^{10} = \frac{e^{x+t}}{2}\left(e^{20}-1\right)$$

$$K_{(3)}(x,t) = \int_0^{10} K(x,y) K_{(2)}(y,t)\,dy = \int_0^{10} e^{x+y}\left(\frac{e^{20}-1}{2}\right) e^{y+t}\,dy$$

$$= e^{x+t}\left(\frac{e^{20}-1}{2}\right)^2$$

$\vdots$

$$K_{(n)}(x,t) = e^{x+t}\left(\frac{e^{20}-1}{2}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$$ $r<1$ yak

$$\Gamma(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_{(n)}(x,t)$$

$$= \sum_{n=1}^{\infty} \lambda^{n-1} e^{x+t}\left(\frac{e^{20}-1}{2}\right)^{n-1} = e^{x+t}\sum_{n=1}^{\infty}\left[\lambda\left(\frac{e^{20}-1}{2}\right)\right]^{n-1}$$

Geometrik seri

$$\frac{\lambda}{2}\left(e^{20-1}\right)<1$$

$$\frac{\lambda}{2}\left(e^{20}-1\right)<1$$

$$= e^{x+t}\,\frac{1}{1-\frac{\lambda}{2}\left(e^{20}-1\right)}$$

$$= \frac{2}{2-\lambda\left(e^{20}-1\right)}\,e^{x+t}$$

$$u(x) = f(x) + \lambda\int_a^b \Gamma(x,t;\lambda)\,f(t)\,dt$$

$$u(x) = e^{-x} + \frac{2\lambda}{2-\lambda\left(e^{20}-1\right)}\int_0^{10} e^{x+t}\,e^{-t}\,dt$$

$$u(x) = e^{-x} + \frac{20\lambda}{2-\lambda\left(e^{20}-1\right)}\,e^{x} \quad \left(\frac{\lambda}{2}\left(e^{20}-1\right)<1\right)$$

Soru

$$u(x) = 1 + \lambda \int_0^1 \frac{2x+1}{(2t+1)^2} u(t)\,dt$$

Cevap : $$u(x) = 1 + \frac{2\lambda(2x+1)}{3(2-\lambda \ln 3)}$$

$$\left| \frac{\lambda \ln 3}{2} \right| < 1$$