Dejenere Çekirdekli Homojen İnt. denklem

$$\boxed{u(x) = \lambda \cdot \int_a^b K(x,t)\,u(t)\,dt} \qquad f(x)=0$$

$$K(x,t) = \sum_{i=1}^{n} r_i(x)\,s_i(t)$$

$u(x)=0$
aşikar çözüm

$$A_i = \int_a^b s_i(t)\,u(t)\,dt \qquad B_i = \int_a^b s_i(t)\,\underbrace{f(t)}_{=0}\,dt = 0$$

$$C_{ij} = \int_a^b s_i(t)\,s_j(t)\,dt$$

$$(1-\lambda C_{11})A_1 - \lambda C_{12}A_2 - \cdots - \lambda C_{1n}A_n = 0$$
$$-\lambda C_{21}A_1 + (1-\lambda C_{22})A_2 - \cdots - \lambda C_{2n}A_n = 0$$
$$\vdots$$
$$-\lambda C_{n1}A_1 - \lambda C_{n2}A_2 - \cdots + (1-\lambda C_{nn})A_n = 0$$

$$\Delta = \begin{vmatrix} 1-\lambda C_{11} & -\lambda C_{12} & \cdots & -\lambda C_{1n} \\ -\lambda C_{21} & 1-\lambda C_{22} & \cdots & -\lambda C_{2n} \\ \vdots & \vdots & & \vdots \\ -\lambda C_{n1} & -\lambda C_{n2} & \cdots & 1-\lambda C_{nn} \end{vmatrix} \qquad A_i$$

$$\Delta = 0$$

$$\Delta = \Delta(\lambda) = 0 \qquad \underbrace{\lambda_1, \lambda_2, \ldots, \lambda_n}_{n \text{ tane}}$$

reel
kompleks

özdeğerler

$\Delta(\lambda) = 0 \qquad \lambda$

$\lambda = \lambda_i \qquad (i = 1, 2, \ldots, n)$

$$\left.\begin{aligned} (1-\lambda_i c_{11})A_1 - \lambda_i c_{12} A_2 - \cdots - \lambda_i c_{1n} A_n &= 0 \\ &\vdots \\ -\lambda_i c_{n1} A_1 - \lambda_i c_{n2} A_2 - \cdots (1-\lambda_i c_{nn}) A_n &= 0 \end{aligned}\right\}$$

$\Delta = 0 \qquad A_1, A_2, \ldots, A_n$ lineer bağımlılık

$$u(x) = \lambda\left[A_1 r_1(x) + A_2 r_2(x) + \cdots + A_n r_n(x)\right]$$

$$\left.\begin{aligned} \lambda = \lambda_1 \text{ için } \quad & u_1(x) \\ \lambda = \lambda_2 \text{ için } \quad & u_2(x) \\ \vdots \\ \lambda = \lambda_n \text{ için } \quad & u_n(x) \end{aligned}\right\}$$ Özfonksiyonlar

Teorem: Özfonksiyonlar ikişer ikişer ortogonaldir.

$$\int_a^b u_i(x) u_j(x)\, dx = 0 \quad i \neq j$$

Örnek $u(x) = \lambda \int_0^1 (x^2 t + x t^2) u(t)\, dt$

$$u(x) = \lambda x^2 \underbrace{\int_0^1 t\, u(t)\, dt}_{A_1} + \lambda x \underbrace{\int_0^1 t^2 u(t)\, dt}_{A_2}$$

$$\boxed{u(x) = \lambda A_1 x^2 + \lambda A_2 x}$$ çözüm modeli

$$\cancel{\lambda} A_1 x^2 + \cancel{\lambda} A_2 x = \cancel{\lambda} \cdot \int_0^1 (x^2 t + x t^2)\left[\lambda A_1 t^2 + \lambda A_2 t\right] dt$$

$$A_1 x^2 + A_2 x \equiv \lambda x^2 \int_0^1 (A_1 t^3 + A_2 t^2)\, dt + \lambda x \int_0^1 (A_1 t^4 + A_2 t^3)\, dt$$

$$* \; A_1 = \lambda \int_0^1 (A_1 t^3 + A_2 t^2)\, dt = \lambda\left(\frac{A_1}{4} + \frac{A_2}{3}\right)$$

$$* \; A_2 = \lambda \int_0^1 (A_1 t^4 + A_2 t^3)\, dt = \lambda\left(\frac{A_1}{5} + \frac{A_2}{4}\right)$$

$$\left.\begin{aligned} \left(1 - \frac{\lambda}{4}\right) A_1 - \frac{\lambda}{3} A_2 &= 0 \\ -\frac{\lambda}{5} A_1 + \left(1 - \frac{\lambda}{4}\right) A_2 &= 0 \end{aligned}\right\} \quad (1)$$

$$\Delta = \begin{vmatrix} 1 - \frac{\lambda}{4} & -\frac{\lambda}{3} \\ -\frac{\lambda}{5} & 1 - \frac{\lambda}{4} \end{vmatrix} = 0$$

$$\left(1 - \frac{\lambda}{4}\right)^2 - \frac{\lambda^2}{15} = 0$$

$$* \; \lambda_1 = \frac{4\sqrt{15}}{4 + \sqrt{15}}$$

$$* \; \lambda_2 = \frac{4\sqrt{15}}{\sqrt{15} - 4}$$

λ_1 değeri (1) de yerine koyup düzenlersek

$$\begin{cases} 3A_1 - \sqrt{15}A_2 = 0 \\ -\sqrt{15}A_1 + 5A_2 = 0 \end{cases}$$

$$\begin{bmatrix} 3 & -\sqrt{15} \\ -\sqrt{15} & 5 \end{bmatrix}$$

$$\underbrace{r = 1 \quad n = 2}$$

$2 - 1 = 1$ tane keyfi

$$3A_1 = \sqrt{15}A_2$$

$$A_1 = \frac{\sqrt{15}}{3} A_2$$

$A_2 = 3$ (keyfi) $\qquad A_1 = \sqrt{15}$

$$u_1(x) = \lambda_1 A_1 x^2 + \lambda_1 A_2 x$$

$$\boxed{u_1(x) = \frac{4\sqrt{15}}{4+\sqrt{15}}\left(\sqrt{15}x^2 + 3x\right)}$$ Özfonk.

Aynı işlemler $\lambda_2 = \dfrac{4\sqrt{15}}{\sqrt{15}-4}$ için yapıldığında

$$\boxed{u_2(x) = \frac{4\sqrt{15}}{\sqrt{15}-4}\left(-\sqrt{15}x^2 + 3x\right)}$$ Özfonk.

$$\int_0^1 u_1(x)u_2(x)\,dx \stackrel{?}{=} 0$$

Resolvant (Çözücü Çekirdek)

$$\boxed{u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt}$$

$$K(x,t) = \sum_{i=1}^{n} r_i(x) s_i(t)$$

dejenere çekirdek

$$u(x) = f(x) + \lambda \left[A_1 r_1(x) + \dots + A_n r_n(x) \right]$$

$$(*)\left\{\begin{array}{l} (1-\lambda C_{11}) A_1 - \lambda C_{12} A_2 \cdots - \lambda C_{1n} A_n = B_1 \\ \quad \vdots \\ -\lambda C_{n1} A_1 - \lambda C_{n2} A_2 \cdots + (1-\lambda C_{nn}) A_n = B_n \end{array}\right.$$

$$A_i = \int_a^b s_i(t)\, u(t)\, dt \qquad B_i = \int_a^b s_i(t) f(t)\, dt \qquad C_{ij} = \int_a^b s_i(t) r_j(t)\, dt$$

(*) sistemi n tane denklem ve n tane (A_i) bilinmeyenden oluşan bir Cramer sistemidir.

i. sütun

Katsayılar det $\to D(\lambda)$

i. sütunlu olan det $\to D_i(\lambda)$

$$A_i = \frac{D_i(\lambda)}{D(\lambda)}$$

$$D_i(\lambda) = \begin{vmatrix} 1-\lambda C_{11} & -\lambda C_{12} \cdots & -\lambda C_{1i-1} & B_1 & -\lambda C_{1i+1} \cdots & -\lambda C_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda C_{n1} & -\lambda C_{n2} \cdots & -\lambda C_{ni-1} & B_n & -\lambda C_{ni+1} & 1-\lambda C_{nn} \end{vmatrix}$$

B_i elemanlarına göre işaretli minörler (eşçarpanlar) Δ_{ji} ile gösterilirse

$$D_i(\lambda) = \sum_{\substack{i=1 \\ j=1}}^{n} B_i \Delta_{ji}$$

$$A_i = \frac{1}{D(\lambda)} \sum_{\substack{i=1 \\ j=1}}^{n} B_i \Delta_{ji} = \frac{1}{D(\lambda)} \sum_{\substack{i=1 \\ j=1}}^{n} \left[\int_a^b s_i(t) f(t)\,dt\right] \Delta_{ji}$$

$$u(x) = f(x) + \lambda \int_a^b \left(\sum_{i=1}^{n} r_i(x) s_i(t) \right) u(t)\,dt$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} A_i r_i(x)$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} r_i(x) \left[\frac{1}{D(\lambda)} \sum_{\substack{i=1 \\ j=1}}^{n} \int_a^b s_i(t) f(t)\,dt \right] \Delta_{ji}$$

$$u(x) = f(x) + \frac{\lambda}{D(\lambda)} \sum_{i=1}^{n} r_i(x) \left(\int_a^b \sum_{\substack{i=1 \\ j=1}}^{n} s_i(t) \Delta_{ji} f(t)\,dt \right)$$

$$u(x) = f(x) + \frac{\lambda}{D(\lambda)} \int_a^b \left[\sum_{i=1}^{n} r_i(x) \left(\sum_{\substack{i=1 \\ j=1}}^{n} s_i(t) \Delta_{ji} \right) \right] f(t)\,dt$$

$$u(x) = f(x) + \lambda \int_a^b \underbrace{\frac{\overbrace{D(x,t;\lambda)}^{D(x,t;\lambda)}}{D(\lambda)}}_{\text{Resolvant}} f(t)\, dt$$

$$\Gamma(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$$

$$u(x) = \underbrace{f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\, dt}$$