# Birinci tür Volterra int. denklemleri

$$f(x) = \int_0^x K(x,t)\,u(t)\,dt$$

$f(x)$

$K(x,t)$ çek. fonk.

$\left.\begin{matrix} u(t) \\ u(x) \end{matrix}\right\}$ Bilinmeyen fonk.

## Seri çözümü yöntemi

$u(x)$ bir analitik fonk. $u(x) = \sum_{n=0}^{\infty} a_n x^n$

$$T(f(x)) = \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$T(f(x)) = \int_0^x K(x,t)\left[a_0 + a_1 t + a_2 t^2 + \cdots\right] dt$$

örnek

$$\underbrace{\sin x - x\cos x}_{} = \int_0^x t\,\underline{u(t)}\,dt \quad \text{seri çözümü yönt.}$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \rightarrow \quad T(\sin x - x\underbrace{\cos x}_{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots}) = \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \cancel{x} + \frac{x^3}{2!} - \frac{x^5}{4!} + \cdots = \int_0^x t\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt$$

$$\frac{x^3}{3} - \frac{x^5}{30} + \frac{x^7}{840} - \cdots = a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3} + a_2 \frac{x^4}{4} + a_3 \frac{x^5}{5} + \cdots$$

$\boxed{a_0 = 0}$; $\quad \frac{a_1}{\cancel{3}} = \frac{1}{\cancel{3}} \Rightarrow \boxed{a_1 = 1}$ ; $\boxed{\cancel{a_2 = 0}}$

$$\frac{a_3}{5} = -\frac{1}{30} \Rightarrow \boxed{a_3 = -\frac{1}{6} = -\frac{1}{3!}}$$

$\boxed{\cancel{a_4 = 0}}$ $\qquad \frac{a_5}{7} = \frac{1}{840} \Rightarrow \boxed{a_5 = \frac{1}{120} = \frac{1}{5!}}$

$$u(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$u(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$\forall x \in \mathbb{R}$ $\quad \boxed{u(x) = \sinh x}$

SORU $\quad 2 + x - 2e^x + xe^x = \int_0^x (x-t)\,u(t)\,dt \quad$ Seri çözümü yapınız.

Cevap: $\boxed{u(x) = xe^x}$

Laplace Dönüşümü Yöntemi ✓

$$f(x) = \int_0^x \underbrace{K(x,t)}\, u(t)\,dt$$

↳ fark çekirdeği

$$f(x) = \int_0^x K(x-t)\,u(t)\,dt$$

$$L\{f(x)\} = L\left\{ \underbrace{\int_0^x \overset{K(x)}{K(x-t)}\,\overset{u(x)}{u(t)}\,dt} \right\}$$

Konvolüsyon teoremi

$L\{f(x)\} = F(s)$ ✓

$L\{K(x)\} = Q(s)$ ✓

$L\{u(x)\} = U(s)$ *

$$F(s) = Q(s)U(s) \longrightarrow U(s) = \frac{F(s)}{Q(s)}$$

$$u(x) = L^{-1}\{U(s)\} = L^{-1}\left\{\frac{F(s)}{Q(s)}\right\} \quad (Q(s) \neq 0)$$

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Örnek $e^{x} - \sin x - \cos x = \int_0^x 2e^{x-t} u(t)\,dt$ Laplace dön. yardımı

$$L\{e^x - \sin x - \cos x\} = L\{\int_0^x 2e^{x-t} \cdot u(t)\,dt\}$$ $L\{u(x)\} = U(s)$

$e^x$ $u(x)$

$$\frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = 2 \cdot \frac{1}{s-1} U(s)$$

$$\frac{\cancel{2}}{(\cancel{s-1})(s^2+1)} = \frac{\cancel{2}}{\cancel{s-1}} U(s)$$

$$U(s) = \frac{1}{s^2+1} \Longrightarrow u(x) = L^{-1}\{U(s)\}$$

$$u(x) = L^{-1}\{\frac{1}{s^2+1}\}$$

$$\boxed{u(x) = \sin x}$$

Örnek $1 + x - e^x = \int_0^x (t-x) u(t)\,dt$

$$L\{1 + x - e^x\} = L\{\int_0^x \underbrace{(t-x)u(t)}_{-(x-t)u(t)}\,dt\}$$ $L\{u(x)\} = U(s)$

$x \cdot u(x)$

$$\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-1} = -\frac{1}{s^2} U(s)$$

$s(s-1)$ $s-1$ $s^2$

$$\frac{\cancel{-}1}{\cancel{s^2}(s-1)} = \frac{\cancel{-}1}{\cancel{s^2}} U(s)$$

$$U(s) = \frac{1}{s-1}$$

$$u(x) = L^{-1}\{U(s)\}$$

$$u(x) = L^{-1}\{\frac{1}{s-1}\}$$

$$\boxed{u(x) = e^x}$$

## 2. tür bir Volterra int. denklemine dönüştürme

Burada 1. tür denklemi 2. tür denkleme dönüştüreceğiz.

Bu dönüşüm, $K(x,x)\neq 0$ ise etkili bir şekilde çalışır.

$$f(x)=\int_0^x \underbrace{K(x,t)u(t)}\,dt$$

Eşitliğin her iki tarafını $x$'e göre türetelim:

$$f'(x)=K(x,x)u(x)+\int_0^x K_x(x,t)u(t)\,dt$$

$$K(x,x)u(x)=f'(x)-\int_0^x K_x(x,t)u(t)\,dt \quad \left(K(x,x)\neq 0\right)$$

$$u(x)=\underbrace{\frac{f'(x)}{K(x,x)}}_{g(x)}-\int_0^x \underbrace{\frac{K_x(x,t)}{K(x,x)}}_{G(x,t)}u(t)\,dt$$

$$u(x)=g(x)+\int_0^x G(x,t)\,u(t)\,dt \rightarrow \text{2. tür Volterra int. denk.}$$

**Örnek** $\sin x=\int_0^x e^{x-t}\cdot u(t)\,dt$ 2. türe dönüştürüp çözünüz.

Her iki tarafı $x$'e göre türetelim:

$$\cos x = u(x)+\int_0^x e^{x-t}\cdot u(t)\,dt$$

$$u(x)=\cos x-\int_0^x e^{x-t}u(t)\,dt \quad \text{2. tür Volterra int.}$$

Laplace ile çözelim:

$$L\{u(x)\} = L\{\cos x\} - L\left\{\int_0^x \underbrace{e^{x-t}}_{e^x} \underbrace{u(t)}_{u(x)} dt\right\} \qquad L\{u(x)\} = U(s)$$

$$U(s) = \frac{s}{s^2+1} - \frac{1}{s-1} U(s)$$

$$\left(1 + \frac{1}{s-1}\right) U(s) = \frac{s}{s^2+1} \rightarrow U(s) \frac{\cancel{s}}{s-1} = \frac{\cancel{s}}{s^2+1}$$

$$U(s) = \frac{s-1}{s^2+1}$$

$$u(x) = L^{-1}\{U(s)\}$$

$$u(x) = L^{-1}\left\{\frac{s-1}{s^2+1}\right\} = L^{-1}\left\{\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$\boxed{u(x) = \cos x - \sin x}$$

örnek

$$1 + \sin x - \cos x = \int_0^x \underbrace{(x - t + 1)}\, u(t)\, dt$$

2. türe çevirelim:

$$\cos x + \sin x = u(x) + \int_0^x u(t)\, dt$$

$$u(x) = \cos x + \sin x - \int_0^x u(t)\, dt \qquad \text{2. tür Volterra int. denk.}$$

Seri çözümü yöntemi ile ;

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_0+a_1 x+a_2 x^2+a_3 x^3+\dots=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots-\int_0^x (a_0+a_1 t+a_2 t^2+\dots)\,dt$$

$$=1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\dots-a_0 x-a_1\frac{x^2}{2}-a_2\frac{x^3}{3}-a_3\frac{x^4}{4}-\dots$$

$$=1+(1-a_0)x-\left(\frac{1+a_1}{2!}\right)x^2-\left(\frac{1}{3!}+\frac{a_2}{3}\right)x^3+\left(\frac{1}{4!}-\frac{a_3}{4}\right)x^4+\dots$$

$a_0=1$

$a_1=1-a_0=1-1=0$

$a_2=-\frac{1+a_1}{2}=-\frac{1}{2!}$

$a_3=-\frac{1}{6}-\frac{a_2}{3}=-\frac{1}{6}+\frac{1}{6}=0$

$a_4=\frac{1}{24}-\frac{a_3}{4}=\frac{1}{24}=\frac{1}{4!}$

$$u(x)=\sum_{n=0}^{\infty}a_n x^n=a_0+a_1 x+a_2 x^2+a_3 x^3+\dots$$

$$u(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$$

$\forall x\in\mathbb{R}$ $\boxed{u(x)=\cos x}$

SORU $\quad 9x^2+5x^3=\int_0^x (10x-10t+6)\,u(t)\,dt$

Cevap: $\boxed{u(x)=3x}$

# Fredholm İnt. Denklemi

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \qquad a, b \in \mathbb{R}$$

## Sabit Çekirdekli İnt. denklemi:

$$u(x) = f(x) + \lambda \int_a^b c \cdot u(t)\, dt \qquad K(x,t) = c \in \mathbb{R}$$

$$\circledast \quad u(x) = f(x) + \lambda c \underbrace{\int_a^b u(t)\, dt} \qquad A = \int_a^b u(t)\, dt \;\checkmark$$

$$\boxed{u(x) = f(x) + \lambda c A}$$

Bu, çözüm için bir model oluşturur. Dolayısıyla int. denklemi sağlamak zorundadır.

$$\cancel{f(x)} + \cancel{\lambda c}A = \cancel{f(x)} + \cancel{\lambda c} \int_a^b [f(t) + \lambda c A]\, dt$$

$$A = \int_a^b f(t)\, dt + \lambda c A \underbrace{\int_a^b dt} \qquad t \Big|_a^b = (b-a)$$

$$A = \int_a^b f(t)\, dt + \lambda c A (b-a)$$

$$A[1 - \lambda c(b-a)] = \int_a^b f(t)\, dt \Rightarrow A = \frac{\int_a^b f(t)\, dt}{1 - \lambda c(b-a)}$$

$$(1 - \lambda c(b-a) \neq 0)$$

$$u(x) = f(x) + \lambda c A$$

$$\Rightarrow \boxed{u(x) = f(x) + \lambda c \frac{\int_a^b f(t)\, dt}{1 - \lambda c(b-a)}}$$

örnek

$$u(x) = x^2 + \lambda \int_0^{10} 3u(t)\,dt \rightarrow * \; u(x) = x^2 + 3\lambda \int_0^{10} u(t)\,dt$$

$$A = \int_0^{10} u(t)\,dt$$

$$\boxed{u(x) = x^2 + 3\lambda A}$$

$$\cancel{x^2} + \cancel{3\lambda} A = \cancel{x^2} + \cancel{3\lambda} \int_0^{10} (t^2 + 3\lambda A)\,dt$$

$$A = \int_0^{10} t^2\,dt + 3\lambda A \int_0^{10} dt$$

$$A = \frac{1000}{3} + 3\lambda A (10 - 0)$$

$$A(1 - 30\lambda) = \frac{1000}{3} \Rightarrow A = \frac{1000}{3(1-30\lambda)} \quad (1 - 30\lambda \neq 0)$$

$$u(x) = x^2 + 3\lambda A$$

$$u(x) = x^2 + \cancel{3}\lambda \frac{1000}{\cancel{3}(1-30\lambda)}$$

$$\boxed{u(x) = x^2 + \frac{1000\lambda}{1-30\lambda}} \quad \boxed{\lambda \neq \frac{1}{30}}$$