YILDIZ TECHNICAL UNIVERSITY, FACULTY OF CIVIL ENG. DEPT. OF GEOMATIC ENG.

ENGINEERING CALCULATIONS

LECTURE NOTES – PART 2

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SPHERICAL TRIGONOMETRY

A sphere is a solid bounded by a surface every point of which is equally distant from a fixed point which is called the centre of the sphere. The straight line which joins any point of the surface with the centre is called a **radius**. A straight line drawn through the centre and terminated both ways by the surface is called a **diameter**.

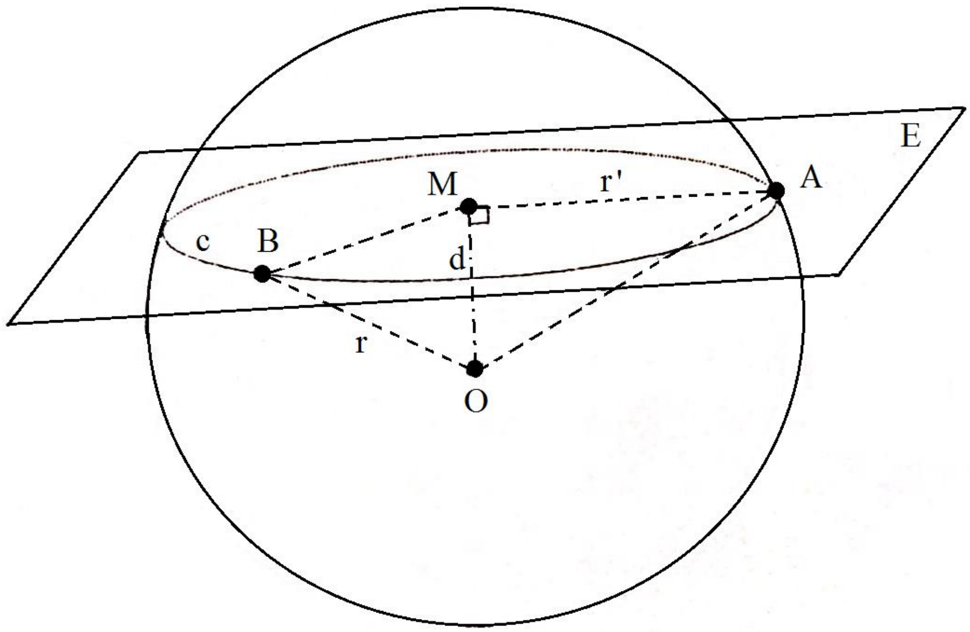


Figure 1. The section of the surface of a sphere made by any plane is a circle.

Let c be the section of the surface of a sphere made by E plane, O the centre of the sphere. Draw OM perpendicular to the plane; take any point A or B in the section and join OB or OA, MA or MB. Since OM is perpendicular to the plane, the angle OMA or OMB is a right angle;

Therefore;

$$MA = \sqrt{(OA)^2 - (OM)^2}$$

Now O and M are fixed points, so that OM is constant; and OA is constant, being the radius of the sphere; hence MA is constant. Thus all points in the plane section are equally distant from the fixed point M; therefore the section is a circle of which M is the centre.

Here;

$OA = OB = r$

$MA = MB = r'$

$OM = d$

$r' = \sqrt{r^2 - d^2}$

The below descriptions can be written for d and r ;

- If $d > r$, then plane E does not intersect with sphere
- If $d = r$, then plane E is tangent to the sphere

- If $d < r$, then plane E intersects with sphere. This circle formed by intersection is so called small circle
- If $d = 0$, then plane E goes through the center of sphere. This circle formed by intersection is so called great circle

GREAT AND SMALL CIRCLES

The section of the surface of a sphere by a plane is called a **great circle** if the plane passes through the centre of the sphere, and a **small circle** if the plane does not pass through the centre of the sphere. **Thus the radius of a great circle is equal to the radius of the sphere**.

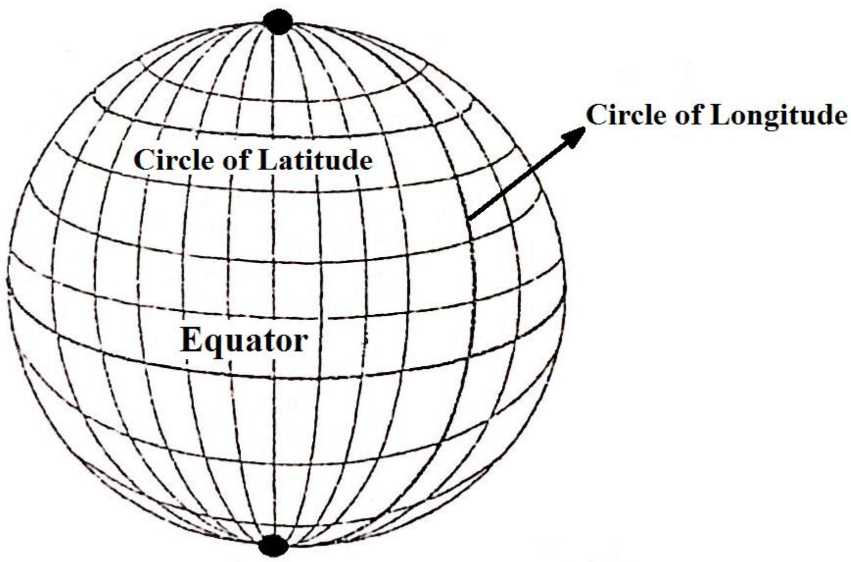


Figure 2. The circles on the sphere

On the globe, equator circle and circles of longitudes are the examples of great circle and, circles of latitudes are the example of small circle.

Latitude: latitude lines are also known as **parallels** (since they are parallel to one another). The lines are actually full circles that extend around the earth and vary in length depending on where we are located. The biggest circle is at the equator and represents the earth's circumference. This line is also called a **Great Circle**. There can be infinitely many great circles, but only one that is a line of latitude (the equator). Any circle that is drawn and is smaller than earth's circumference is called a **Small Circle**.

Longitude: longitude lines are also known as **meridians**. The lines extend in a N-S direction, but are used to state locational positions either east or west of a set location. This location is known as the **Prime Meridian**. Each meridian is exactly half of a great circle. Meridians are not parallel - the spacing between them is the greatest at the equator (111.2 km) and decreases to zero at both poles. Meridians intersect the parallels at right angles.

The Prime Meridian runs from the North Pole through Greenwich, England (the Royal Observatory outside of London) and to the South Pole. From this location, west longitude values increase from 0° to 180° halfway around the earth. Same applies to the east.

A plane intersecting the globe along a great circle divides the globe into equal halves and passes through its center.

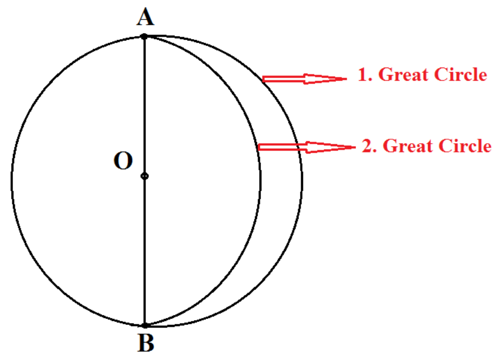


Figure 3. The great circles

A plane that intersects the globe along a small circle splits the globe into unequal sections. This plane does not pass through the center of globe.

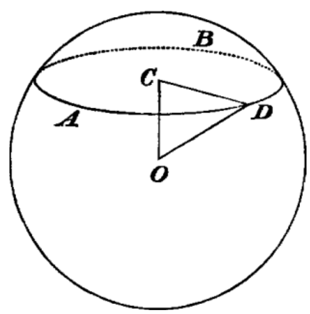


Figure 4. The small circle

Through the centre of a sphere and any two points on the surface a plane can be drawn; and only one plane can be drawn, except when the two points are the extremities of a diameter of the sphere, and then an infinite number of such planes can be drawn. Hence **only one great circle can be drawn through two given points on the surface of a sphere, except when the points are the extremities of a diameter of the sphere**.

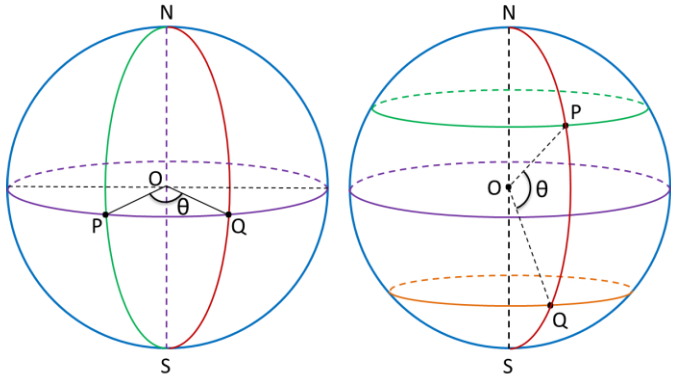


Figure 5. Through the centre of a sphere and any two points on the surface only a plane can be drawn

If these mentioned two given points are the end points of a diameter, then infinite number of great circle can be drawn.

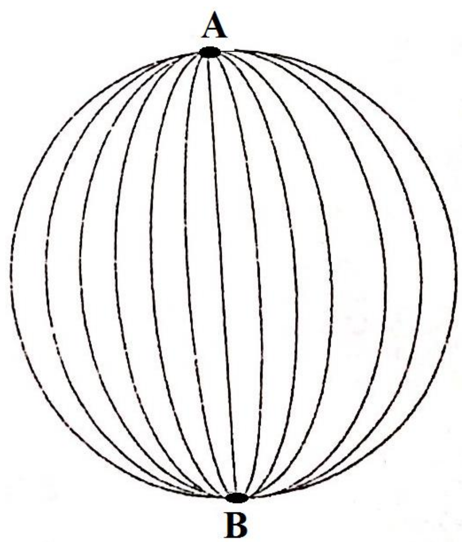
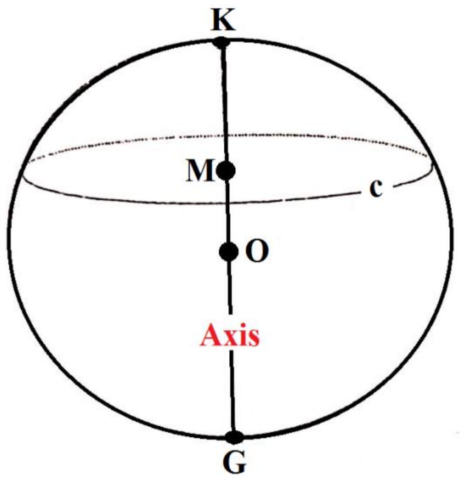


Figure 6. The great circles

The great circle is an imagery line that follows the curve of the earth and represents the shortest distance between two points. The spherical distance between two points **P** and **Q** on a sphere is the distance of the shortest path along the surface of the sphere

The **axis of any circle of a sphere** is that **diameter of the sphere** which is perpendicular to the plane of the circle; the extremities of the axis are called the **poles of the circle**. The poles of a great circle are equally distant from the plane of the circle. The poles of a small circle are not equally distant from the plane of the circle; they may be called respectively the nearer and further pole; sometimes the nearer pole is for brevity called the pole.



The diameter of the circle namely KG is the axis of circle c, and K and G are the poles of this circle (c).

Figure 7. The axis of circle

The axis of equator which is intersection to a plane passing through the center of the earth is called as rotation axis, which is perpendicular to this circle.

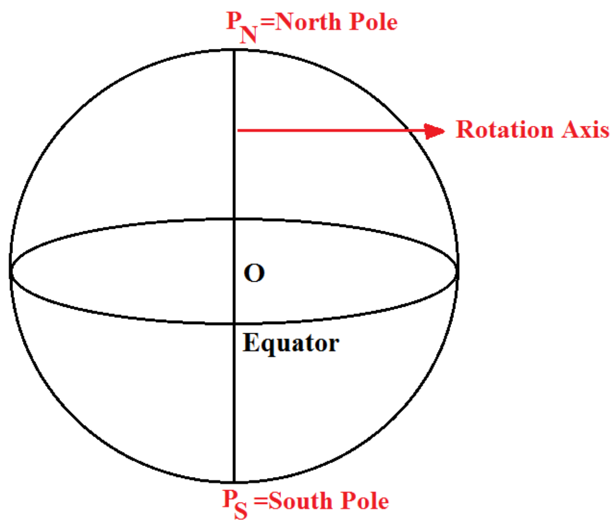


Figure 8. The Poles and rotation axis

Extremities of the rotation axis on the globe are called as North Pole and South Pole. *A **pole of a circle** is equally distant from every point of the circumference of the circle.*

*The distance between two points on the globe is measured by the arc of great circle passing through these two points. This arc is selected as no bigger than half of the great circle. The distance defined as mentioned above is called as **spherical distance**. At the below illustration, the bold line represents the spherical distance between A and B.*

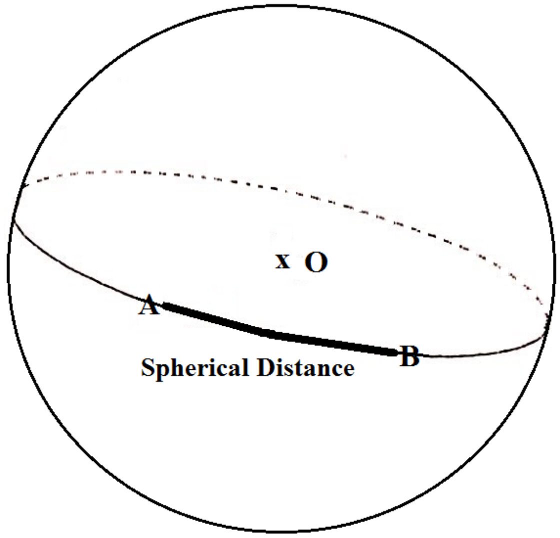


Figure 9. Spherical Distance

To show that the shortest distance between A and B is the great circle:

Let O be the centre of the sphere, A and B be any two points on the sphere. Infinite number of plane can be drawn with A and B. the shortest distanced arc is the intersection of one of those planes and sphere. Thus, infinite number of sphere arc can be drawn with these two points. Since, if, the small circle is rounded on AB and the great circle is intersected with it, the below illustration is obtained.

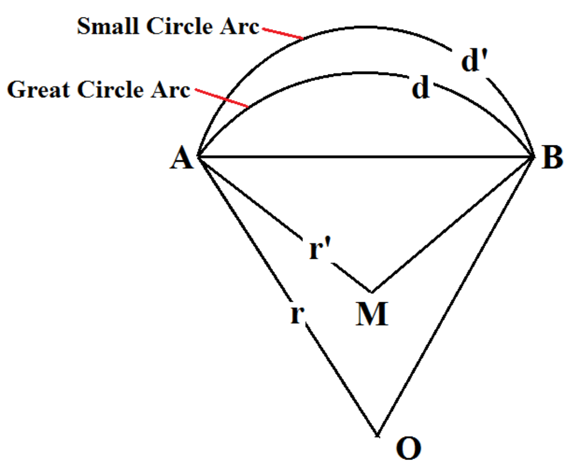


Figure 10. The shortest distance on the sphere

If, $r' < r$; then $d' > d$

The shortest arc distance drawn between A and B, has **the smallest curvature**, which means, it has **the largest radius**. The largest arc of a sphere is the great circle, thus, the largest arc passing through these two points is smaller than the small circle.

By the angle between two great circles is meant *the angle of inclination of the planes of the circles*. Thus, in the figure, the angle between the great circles PA and PB is the angle AOB.

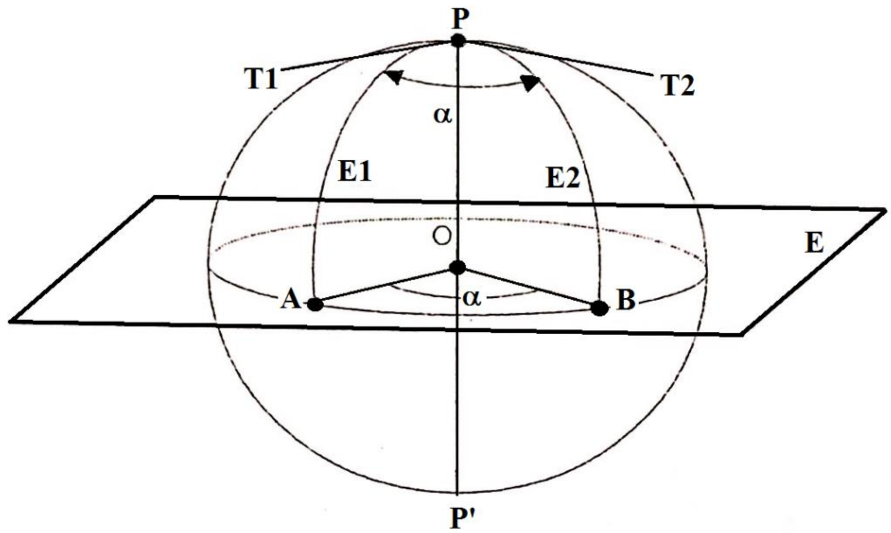


Figure 11. The angle of inclination of the planes of the circles

Angle APB is equal to angle of inclination of planes E1 and E2. In other words, the angle between tangents of PT1 and PT2 is spherical angle. It can be measured by either angle α or arc AB.

SPHERICAL DISTANCE

If two points on the sphere are connected with a great circle, this interval can be defined as either angle or distance. At the following figure, spherical distance y between A and B subtends to angle α at the center of the sphere.

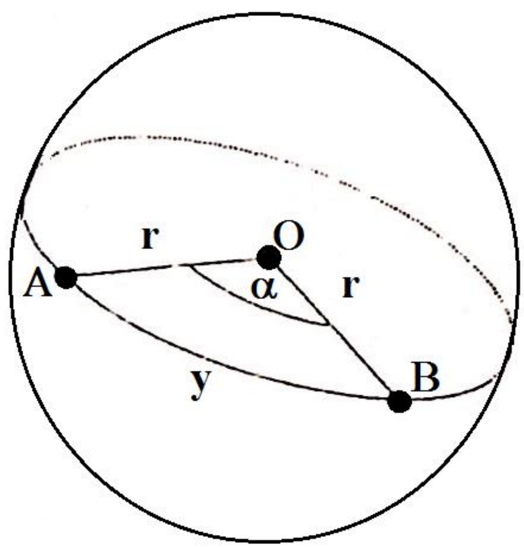


Figure 12. The spherical distance

Let r be the radius of sphere, then the equation between y and α can be written as;

$$y = \alpha . r$$

Here; the unit of α is radian. To obtain the value in grad or degree, it should be divided to ρ:

$$y = \frac{\alpha}{\rho} . r$$

If the distance is known, then the angle corresponded to it can be calculated as:

$$\alpha = \frac{y}{r} . \rho$$

The spherical distance is generally defined as angle unit. For instance; to compute the arc distance of 1', let the radius be 6370000 m,

$$y = \frac{\alpha}{\rho} . r = \frac{1}{3437.74677} * 6370000 = 1852.96 m$$

This value is so called as nautical mile/geographical mile.

SPHERICAL SHAPES

Spherical Lune

Two great circles on the sphere divide the sphere into four pieces. Each piece is so called spherical lune. In other words, the spherical lune is the surface area of a sphere between two planes which intersect at the diameter. Moreover, these spherical lunes are equal to each other correspondingly.

Since the area of the whole sphere is $4\pi r^2$, then Fa shows the area of the spherical lune defined by angle α;

$$F_a = \frac{4.\pi.r^2}{360^o} . \alpha^o = \frac{2\pi.r^2}{180^o} . \alpha^o$$

By re-arranging the equation:

$$F_a = \frac{2.r^2}{\frac{180^o}{\pi}} . \alpha^o$$

By using the abbreviation of ρ

$$F_a = 2.r^2 \frac{\alpha^o}{\rho^o}$$

$$\widehat{\alpha} = \frac{\alpha^o}{\rho^o}$$

$$F_a = 2.r^2.\widehat{\alpha}$$

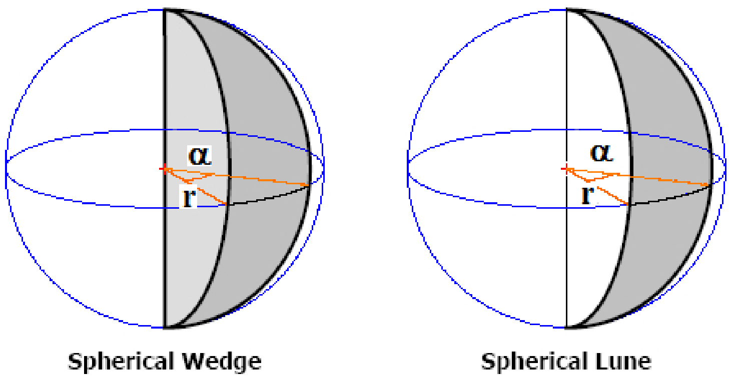


Figure 13. The spherical wedge and spherical lune

Spherical Wedge

$$V_a = \frac{V}{360^o}.\alpha^o$$

The volume of whole sphere, $V = \frac{4}{3}.\pi.r^3$ and the volume of the spherical wedge is given as below:

$$V_a = \frac{\frac{4}{3}.\pi.r^3}{360^o}.\alpha^o$$

$$V_a = \frac{2}{3}.\widehat{\alpha}.r^3$$

Spherical Cap

A spherical cap is the region of a sphere which lies above (or below) a given plane. If the plane passes through the center of the sphere, the cap is a called a hemisphere.

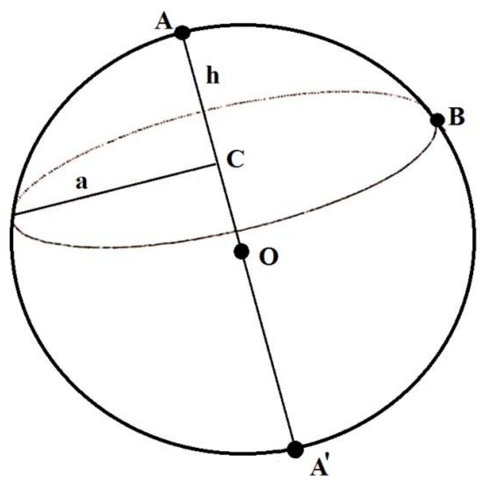


Figure 14. The spherical cap

Let C be the center of the circle drawn by B; r be the radius of sphere, and AC=h, the area of the spherical cap surface is computed with the below equation.

$$F_{SC} = 2.\pi.r.h$$

As mentioned above, since the sphere is divided into two pieces by a given plane, the volume of smaller cap is computed as;

$$V = \frac{\pi}{3}.h^2.(3.r - h)$$

If the volume is computed by the radius of circle (a), then:

$$V = \frac{\pi}{6}.h.(3.a^2 + h^2)$$

Spherical Segment

A spherical segment is the solid defined by cutting a sphere with a pair of parallel planes. It can be thought of as a spherical cap with the top truncated, and so it corresponds to a spherical frustum. The surface of the spherical segment (excluding the bases) is called a zone.

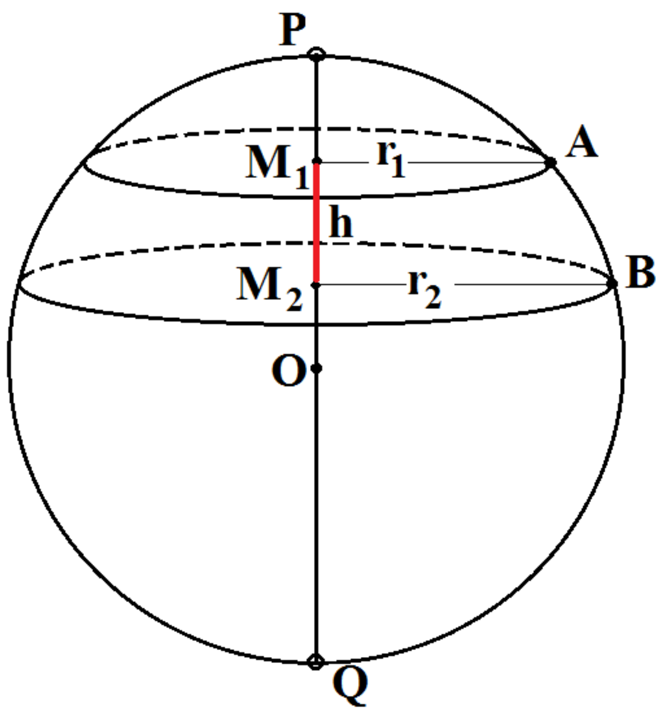


Figure 15. The spherical segment

Let r_1 be the radius of the circle drawn by A, r_2 be the radius of the circle drawn by B, r be the radius of sphere, h be the distance between M_1 and M_2,

The area of the spherical segment is computed as below:

$$F = F_2 - F_1$$

$$F_2 = 2.\pi.r.h_2$$

$$F_1 = 2.\pi.r.h_1$$

$$F = 2.\pi.r.h_2 - 2.\pi.r.h_1$$

$$F = 2.\pi.r.(h_2 - h_1)$$

$$F = 2.\pi.r.h$$

The volume of the spherical segment is computed as below:

$$V = \frac{\pi}{6}.h.(3r_1^2 + 3r_2^2 + h^2)$$

SPHERICAL TRIANGLES

Spherical Trigonometry investigates the relations which subsist between the angles of the plane faces which form a solid angle and the angles at which the plane faces are inclined to each other.

Suppose that the angular point of a solid angle is made the centre of a sphere; then the planes which form the solid angle will cut the sphere in arcs of great circles. Thus a figure will be formed on the surface of the sphere which is called a **spherical triangle** if it is bounded by **three arcs of great circles**; this will be the case when the solid angle is formed by the meeting of three plane angles. If the solid angle be formed by the meeting of more than three plane angles, the corresponding figure on the surface of the sphere is bounded by more than three arcs of great circles, and is called a spherical polygon.

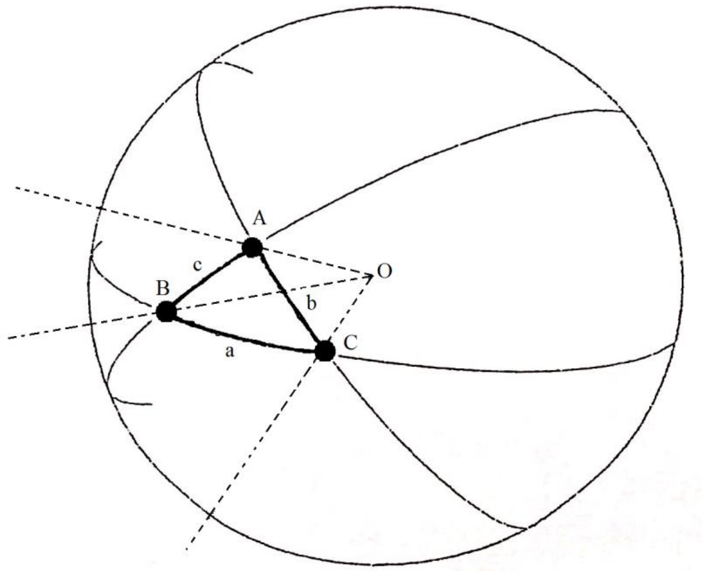


Figure 16. Spherical Triangle and its elements

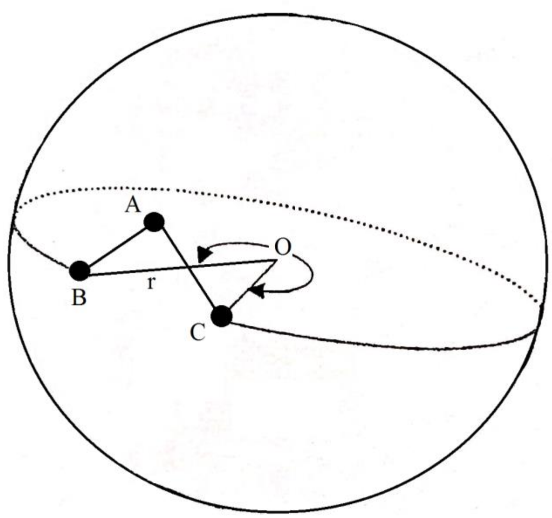


Figure 17. General Spherical Triangle

The length of arc BC;

$$BC = r.B\hat{O}C$$

Here; the unit of angle $B\hat{O}C$ is radian. Similarly, for the arc lengths of AB and AC;

$$AB = r.A\hat{O}B$$
$$AC = r.A\hat{O}C$$

As seen from the formulas, since the same *r* (radius of sphere) is used to compute the lengths of every great circle, r should be taken as unit. Thus, **the length of a great circle is equal to the central angle that subtends the arc from the center**. Therefore, sides of the great circles can be defined as angles.

The computations and formulas given for the great circle cannot be implemented to solve a small circle.

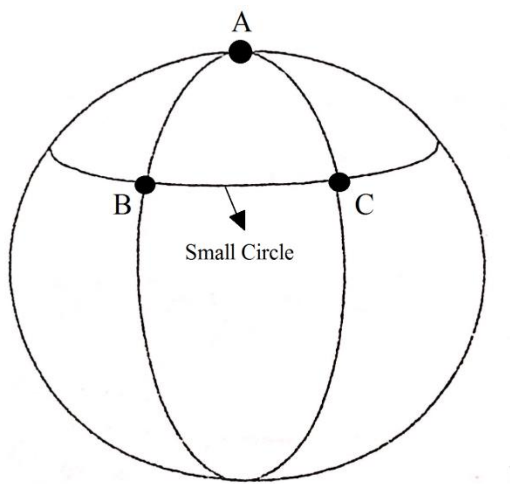


Figure 18. ABC is not spherical triangle

If any sides of the circle are no bigger than 180 degree, this circle is so called **Euler spherical triangle**.

The three arcs of great circles which form a spherical triangle are called the **sides of the spherical triangle**; the angles formed by the arcs at the points where they meet are called the **angles of the spherical triangle**.

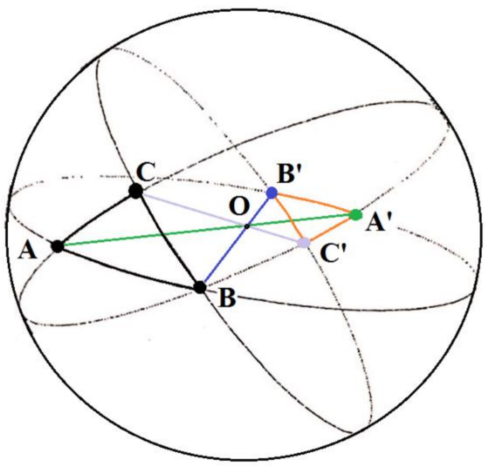
Let ABC be any spherical triangle, and let the points A', B', C' be those poles of the arcs BC, CA, AB respectively which lie on the same sides of them as the opposite angles A, B, C; then the triangle $A'B'C'$ is said to be the polar triangle of the triangle ABC.

Since there are two poles for each side of a spherical triangle, eight triangles can be formed having for their angular points poles of the sides of the given triangle; but there is only one triangle in which these poles $A'B'C'$ lie towards the same parts with the corresponding angles A, B, C; and this is the triangle which is known under the name of the polar triangle. The triangle ABC is called the primitive triangle with respect to the triangle $A'B'C'$.

If three great circles are drawn from points A, B, C among AB, BC, and CA; let O be the center of a sphere, and suppose a solid angle formed at O by the meeting of three plane angles. Let AB, BC, CA be the arcs of great circles in which the planes cut the sphere; then ABC is a spherical triangle, and the arcs AB, BC, CA are its sides. Three intersection points are found symmetric to center of sphere O. In this case, Euler Spherical Triangles are formed. Let A', B' and C' denote the intersection points of the great circles from the sphere center.

The 8 Euler Spherical Triangles can be formed as:

ABC ; $A'BC$; $AB'C$; ABC'
$A'B'C'$; $AB'C'$; $A'BC'$; $A'B'C$



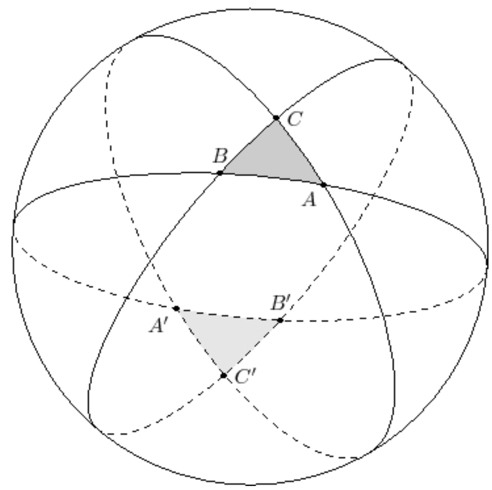


Figure 19. The eight spherical triangles

The triangles, which are symmetric to the center of O and equal to each other, are so called **polar triangle.**

Area of a Spherical Triangle & Spherical Excess

To compute the area of ABC triangle, equations of spherical lune are used.

To find the area of a Lune:
A Lune is that portion of the surface of a sphere which is comprised between two great semicircles.

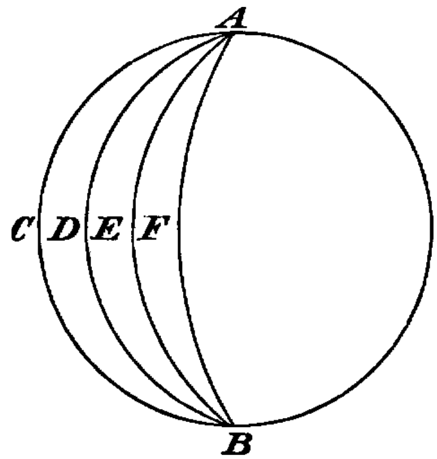


Figure 20. The spherical lune

Let ACBDA, ADBEA be two lunes having equal angles at A; then one of these lunes may be supposed placed on the other so as to coincide exactly with it; thus lunes having equal angles are equal. Then it may be shown that lunes are proportional to their angles. Hence since the whole surface of a sphere may be considered as a lune with an angle equal to four right angles, we have for a lune with an angle of which the circular measure is A,

$$\frac{Area\ of\ Lune}{Surface\ of\ Sphere} = \frac{A}{2.\pi}$$

Suppose r the radius of the sphere, then the surface is $4.\pi.r^2$;

Area of lune: $\frac{A}{2.\pi}.4.\pi.r^2 = 2.A.r^2$

To find the area of a Spherical Triangle:

Let ABC be a spherical triangle; produce the arcs which form its sides until they meet again two and two, which will happen when each has become equal to the semicircumference.

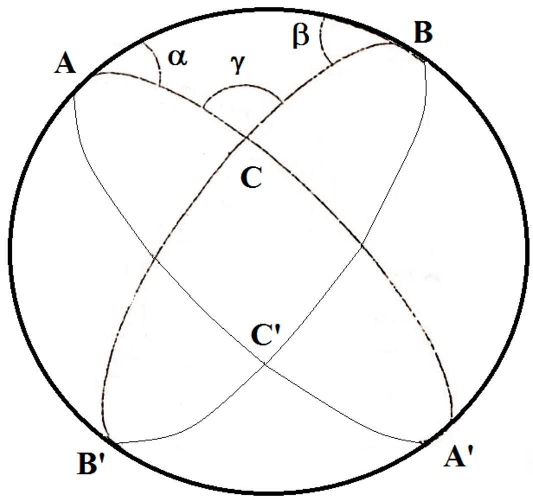


Figure 21. The area of spherical triangle

The triangle ABC now forms a part of three lunes, namely, $ABA'CA$, $BCB'AB$, and $CAC'BC$. Now the triangles $CA'B'$ and $AC'B$ are subtended by vertically opposite solid angles at O, and we will assume that their areas are equal; therefore the lune $CAC'BC$ is equal to the sum of the two triangles ABC and $CA'B'$. Hence if A, B, C denote the circular measures of the angles of the triangle, we have:

$$F_{ABC} + F_{A'BC} = F_{\alpha} = 2.\widehat{\alpha}.r^2$$
$$F_{ABC} + F_{AB'C} = F_{\beta} = 2.\widehat{\beta}.r^2$$
$$F_{ABC} + F_{ABC'} = F_{\gamma} = 2.\widehat{\gamma}.r^2$$

Hence, by addition:

$$3.F_{ABC} + F_{A'BC} + F_{AB'C} + F_{ABC'} = 2.r^2.(\widehat{\alpha} + \widehat{\beta} + \widehat{\gamma})$$

From the above illustration,

$$F_{A'BC} + F_{AB'C} + F_{ABC'} = 2.\pi.r^2 - F_{ABC}$$

By re-arranging the equation:

$$2.F_{ABC} + 2.\pi.r^2 = 2.r^2.(\widehat{\alpha} + \widehat{\beta} + \widehat{\gamma})$$

Therefore:

$$F_{ABC} = r^2.(\widehat{\alpha} + \widehat{\beta} + \widehat{\gamma} - \pi)$$

If the angles are given in degrees:

$$F_{ABC} = \frac{\alpha + \beta + \gamma - 180^o}{\rho^o}.r^2$$

The expression $(\alpha + \beta + \gamma - 180^o)$ is called the spherical excess of the triangle and denote by ε;
$\varepsilon = (\alpha + \beta + \gamma - 180^o)$

$$F_{ABC} = \frac{\varepsilon}{\rho}.r^2$$

Hence, if the equation is re-arranged as to ε;

$$\varepsilon = \frac{F_{ABC}}{r^2}.\rho$$

The sum of three angles of the spherical triangle is not constant, and is proportional to the area of spherical triangle. Since area is getting bigger, the excess is getting bigger as well. Just the contrary is also valid similarly. If the radius of a sphere is infinite, then it will be a plane. Thus, the excess will be 0 and the sum of the angles will be 180^o. The area of small spherical triangles can be supposed to the plane triangles which have the same side lengths. These triangles are assumed as small spherical triangles when considering their sides are so small comparing with the radius of sphere. For instance, for the triangles having the sides up to 100 km are assumed as this.

Polar Triangle and Trihedron

The triangles, which are symmetric to the center of O and equal to each other, are so called **polar triangle.**

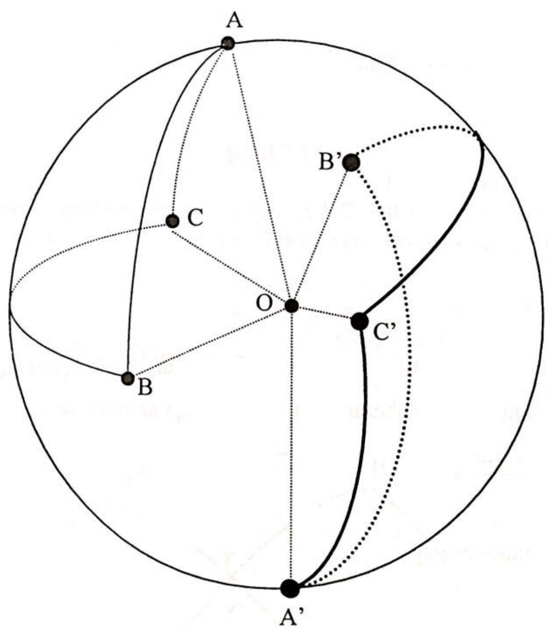


Figure 22. The polar triangle

Theorem: Let ABC be any spherical triangle, and let the points A', B', C' be those poles of the arcs BC, CA, AB respectively which lie on the same sides of them as the opposite angles A, B, C; then the triangle $A'B'C'$ is said to be the polar triangle of the triangle ABC.

Theorem: The angles of the spherical triangle are the supplementary of the sides of the polar triangle. Or; the sides of the spherical triangle are the supplementary of the angles of the polar triangle.

if A, B, C, a, b, c denote respectively the angles and the sides of a spherical triangle, all expressed in circular measure, and A', B', C', a', b', c' those of the polar triangle

$$A + a' = 180^o \qquad a + A' = 180^o$$
$$B + b' = 180^o \qquad b + B' = 180^o$$
$$C + c' = 180^o \qquad c + C' = 180^o$$

If three corners of a spherical triangle are connected to the center of this sphere, a trihedron is formed. O –center of the sphere- is the apex; OA, OB and OC are the sides; AOB, BOC and COA are the faces of the trihedron. Since any spherical triangle has a trihedron, reverse is valid as well. Angles between the sides of the trihedron are sides of the spherical triangle.

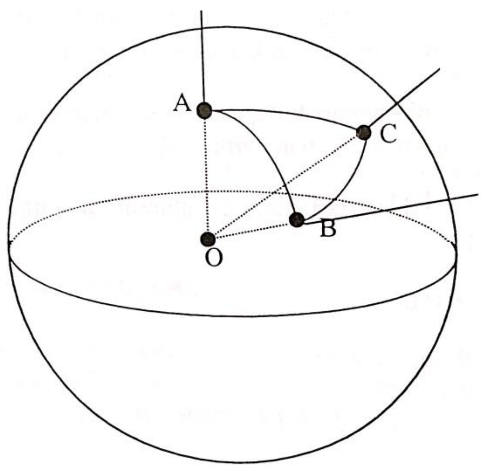


Figure 23. The trihedron

Specifications of Spherical Triangles

The letters A, B, C are used to denote the angles of a spherical triangle, and the letters a, b, c are used to denote the sides.

1. **Theorem:** The sum of the angles of the spherical triangles is between 180 and 540 degrees, and the sum of its sides is between 0 and 360 degrees.
   For a ABC spherical triangle:

$$180^o < A+B+C < 540^o$$
$$0^o < a+b+c < 360^o$$

Proof: On the sphere, the biggest spherical triangle is a little smaller than the hemisphere. Let the area of the hemisphere be $2.\pi.r^2$; then

$$F = r^2.(\widehat{\alpha}+\widehat{\beta}+\widehat{\gamma}-\pi)$$
$$2.\pi.r^2 = r^2.(A+B+C-\pi)$$
$$A+B+C = 3.\pi = 540^o$$

Since a hemisphere is not a spherical triangle, it cannot involve this value.
Moreover, to have a positive area value; the expression should be written as below.

$$A+B+C-\pi > 0^o$$
$$A+B+C > 180^o$$

To find the limits of the sum of the sides; polar triangle can be used:

$$a + A' = 180^o$$
$$b + B' = 180^o$$
$$c + C' = 180^o$$

By addition:

$$A'+B'+C'+a+b+c = 540^o$$

The polar triangle is also a spherical triangle, and 180^o and 540^o, which are limits of angles can be written instead of $(A'+B'+C')$. Then, the following expression is obtained

$$0^o < a+b+c < 360^o$$

2. **Theorem:** For any two of the three plane angles which form the solid angle at O are together greater than the third. Therefore any two of the arcs AB, BC, CA, are together greater than the third side. From this proposition it is obvious that any side of a spherical triangle is greater than the difference between the other two sides.

$$\left.\begin{matrix} a+b>c \\ b+c>a \\ a+c>b \end{matrix}\right\} \quad \text{Or} \quad \left.\right\} \begin{matrix} c-a<b \\ a-b<c \\ b-c<a \end{matrix}$$

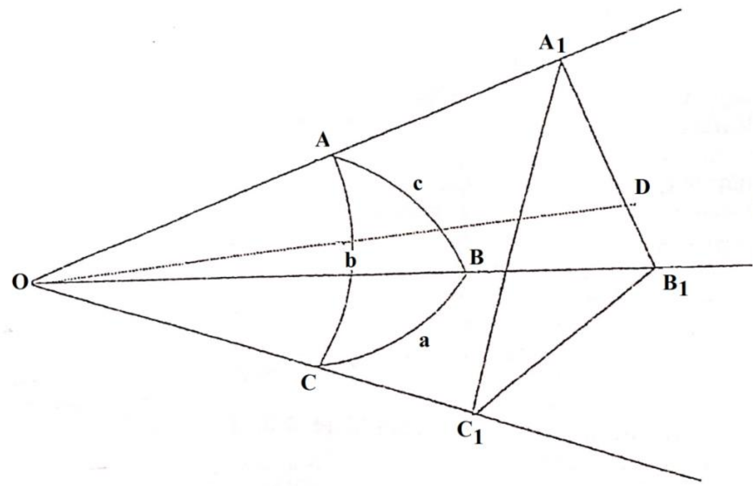


Figure 24. The trihedron

Proof: For a OABC trihedron, let the relation between two sides be $c > b$. Let A1 be on OA side, and B1 be on OB side. Consider that a line is drawn on AOB face starting from OA line with the same angle value of AOC. Let D1 be the intersection point of this line. Let C1 be taken on OC line as OD=OC1.
On the newly formed A1B1C1 plane triangle;

$A_1C_1 + C_1B_1 > A_1D + DB_1$
The triangles A_1OD_1 *and* B_1OD are equal to each other. $A_1OD_1 = B_1OD$
Thus, $A_1C_1 = A_1D$
Then, $C_1B_1 > DB_1$

Since The OB_1 side is common for ODB_1 *and* OB_1C_1 triangles, $OC_1 = OD$, and considering the above inequation, the below can be written

$B_1OC_1 > B_1OD$

Extending the inequation adding A_1OC_1 to the left side, and A_1OD to the right side:

$A_1\hat{O}C_1 + B_1\hat{O}C_1 > B_1\hat{O}D + A_1\hat{O}D$
Or:
$A\hat{O}C + B\hat{O}C > A\hat{O}B$

If the arcs are written for these angles, which are subtended to them; then

$a + b > c$
$c - a < b$

3. **Theorem:** For any two of the three plane angles which form the solid angle at O are smaller than the third one by adding 180 degrees.

$$A + B < 180^o + C$$
$$B + C < 180^o + A$$
$$C + A < 180^o + B$$

Proof: If the relationship between spherical triangle and polar triangle are implemented to the inequations noted at 2nd specifications,

$$a + b > c$$
$$180^o - A' + 180^o - B' > 180^o - C'$$

By re-arranging;

$$A' + B' < 180^o + C'$$

4. **Theorem:** The angles at the base of an isosceles spherical triangle are equal.

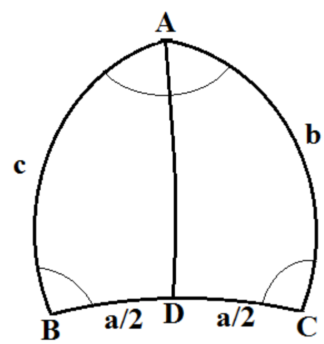


Figure 25. The isosceles triangle

If; $b = c$; then $B = C$
Or
If; $B = C$; then $b = c$

Proof: Let ABC be a spherical triangle having AB =AC; let O be the center of the sphere. Let D be the midpoint of arc BC. Then, the spherical triangles ABD and ACD are symmetrically equal to each other. Thus, $B = C$.
If $b = c = 90^o$, A will be the pole of the great circle of BC, and moreover $B = C = 90^o$ and $A = a$

Since $B = C$; by using the polar triangle specifications given below;

$$B + b' = 180^o$$
$$C + c' = 180^o$$

It can be noted that: $b' = c'$, which means this polar spherical triangle is also an isosceles spherical triangle. It can be also expressed by $B' = C'$.
$b = c$

5. **Theorem:** If one angle of a spherical triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less angle.

   If $B > C$; then $b > c$;
   Or:
   If $b > c$; then $B > C$

   Proof: Let ABC be a spherical triangle having $B > C$. At B make the angle CBD equal to the angle BCD; Let D the intersection point on arc AC. The spherical triangle divides into two namely, ABD and DBC.

   Using the second specification for ABD triangle,
   $BD + DA > AB$

   Then, from the forth specification; BD is equal to DC; BD=DC
   Applying this equation to the $BD + DA > AB$;
   $CD + DA > AB$ or $b > c$

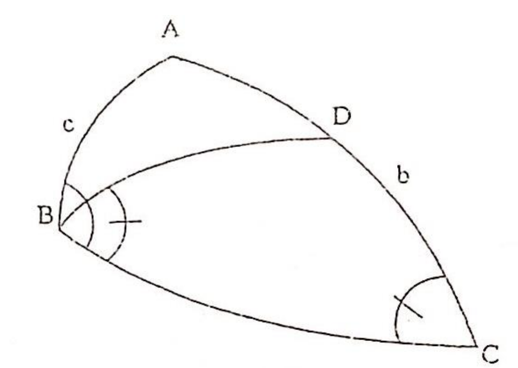


Figure 26a. The spherical triangle

Reversely; $b > c$; take c on AC arc; then:

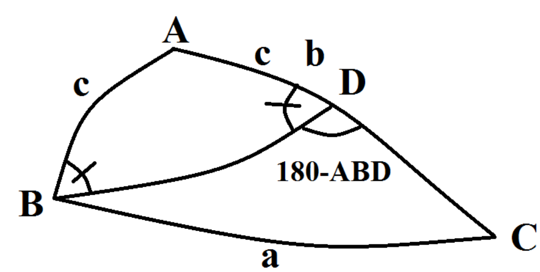


Figure 26b. The spherical triangle

$ABD = ADB$
$BDC + C < DBC + 180^o$
Here, let BDC be $180^o - ABD$, then
$180^o - ABD + C < DBC + 180^o$
By arranging;
$ABD + DBC > C \Rightarrow B > C$

6. Theorem: For a spherical triangle;

If;
$\ldots\ldots\ldots = \ldots\ldots$ $\qquad$ $\ldots\ldots\ldots = \ldots\ldots$
$a + b > 180^o$ then, $A + B > 180^o$
$\ldots\ldots\ldots < \ldots\ldots$ $\qquad$ $\ldots\ldots\ldots < \ldots\ldots$

Or;

If;
$\ldots\ldots\ldots = \ldots\ldots$ $\qquad$ $\ldots\ldots\ldots = \ldots\ldots$
$A + B > 180^o$ then, $a + b > 180^o$
$\ldots\ldots\ldots < \ldots\ldots$ $\qquad$ $\ldots\ldots\ldots < \ldots\ldots$

Proof: In case of $a + b = 180^o$, on the arc AC starting from C, extend the arc up to a. The last point of the extended arc is called as A'. The newly formed shape is a spherical lune. Thus A=A'. the A'BC is an isosceles spherical triangle and the bottom angles are equal to each other.
Thus,

$A' = C\hat{B}A' = A$

If the equation $C\hat{B}A' = 180^o - B$ is implemented to the above,
$A = 180^o - B$ or $A + B = 180^o$ is obtained.

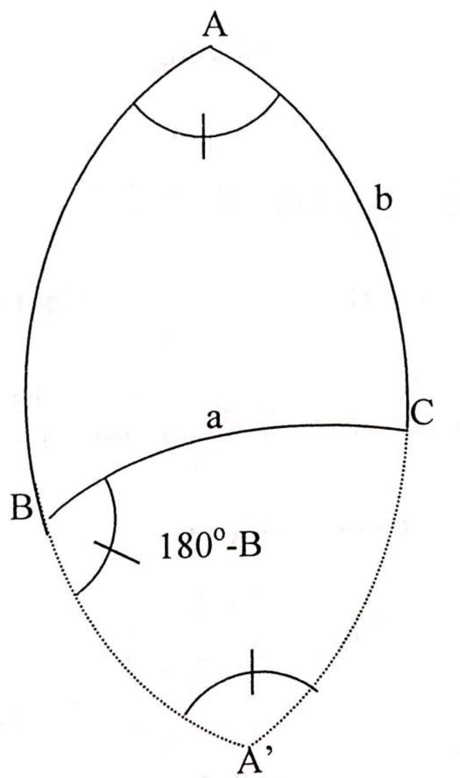


Figure 27. The spherical triangle

If $a + b < 180^o$,
$CA' > BC$

7. **Theorem:** For a spherical triangle;

$$-90^o < \frac{-A + B + C}{2} < 90^o$$
$$-90^o < \frac{A - B + C}{2} < 90^o$$
$$-90^o < \frac{A + B - C}{2} < 90^o$$

Proof: Using the 3rd specification;
$B + C - A < 180^o$
According to $B + C > A$ or $B + C < A$; the below can be written:
$-180^o < B + C - A < 180^o$

THEOREMS REGARDING SPHERICAL TRIANGLE

SINE THEOREM

The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides and proportion is constant.

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=M=constant$$

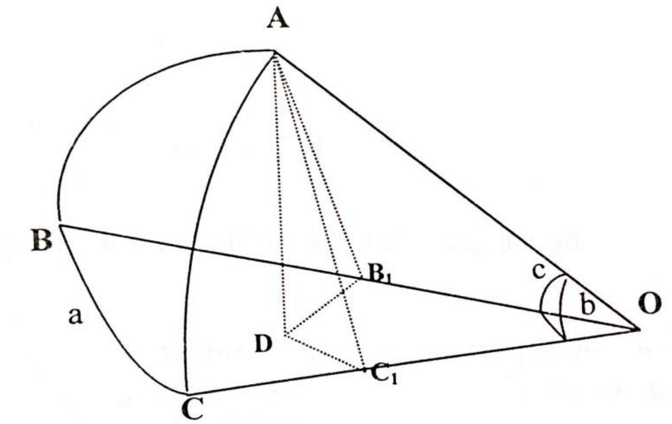


Figure 28. The sine theorem

COSINE of a SIDE THEOREM

To express the cosine of a side of a triangle in terms of sines and cosines of the angles:

$$\begin{aligned}\cos a&=\cos b.\cos c+\sin b.\sin c.\cos A\\ \cos b&=\cos a.\cos c+\sin a.\sin c.\cos B\\ \cos c&=\cos a.\cos b+\sin a.\sin b.\cos C\end{aligned}$$

Reminder: In the plane geometry;
$a^2=b^2+c^2-2.b.c.\cos A$

COSINE of an ANGLE THEOREM

To express the cosine of a side of a triangle in terms of sines and cosines of the angles:

$$\begin{aligned}\cos A&=-\cos B.\cos C+\sin B.\sin C.\cos a\\ \cos B&=-\cos A.\cos C+\sin A.\sin C.\cos b\\ \cos C&=-\cos A.\cos B+\sin A.\sin B.\cos c\end{aligned}$$

SINE - COSINE THEOREM

$$\sin a.\cos B = \cos b.\sin c - \sin b.\cos c.\cos A$$
$$\sin b.\cos C = \cos c.\sin a - \sin c.\cos a.\cos B$$
$$\sin c.\cos A = \cos a.\sin b - \sin a.\cos b.\cos C$$

$$\sin a.\cos C = \cos c.\sin b - \sin c.\cos b.\cos A$$
$$\sin b.\cos A = \cos a.\sin c - \sin a.\cos c.\cos B$$
$$\sin c.\cos B = \cos b.\sin a - \sin b.\cos a.\cos C$$

$$\sin A.\cos b = \cos B.\sin C + \sin B.\cos C.\cos a$$
$$\sin B.\cos c = \cos C.\sin A + \sin C.\cos A.\cos b$$
$$\sin C.\cos a = \cos A.\sin B + \sin A.\cos B.\cos c$$

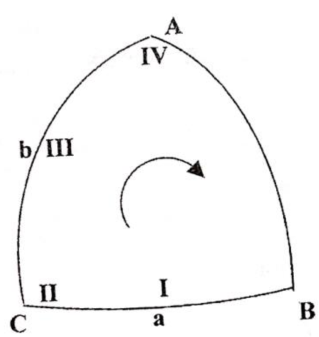
$$\sin A.\cos c = \cos C.\sin B + \sin C.\cos B.\cos a$$
$$\sin B.\cos a = \cos A.\sin C + \sin A.\cos C.\cos b$$
$$\sin C.\cos b = \cos B.\sin A + \sin B.\cos A.\cos c$$

COTANGENT THEOREM

$$\cos c.\cos A = \sin c.\cot b - \sin A.\cot B$$
$$\cos a.\cos B = \sin a.\cot c - \sin B.\cot C$$
$$\cos b.\cos C = \sin b.\cot a - \sin C.\cot A$$

$$\cos b.\cos A = \sin b.\cot c - \sin A.\cot C$$
$$\cos c.\cos B = \sin c.\cot a - \sin B.\cot A$$
$$\cos a.\cos C = \sin a.\cot b - \sin C.\cot B$$

To generalize the formulas given above; the numbered spherical triangle may be used as follow as starting from side element:



$$\cos III.\cos II = \sin III.\cot I - \sin II.\cot IV$$

Other Spherical Triangle Formulas

Half-Angle Formulas:

According to Cosine of a side theorem;

$$\cos a = \cos b.\cos c + \sin b.\sin c.\cos A$$

Put cosA to the left side of the equation,

$$\cos A = \frac{\cos a - \cos b.\cos c}{\sin b.\sin c}$$

Subtract both sides from 1 of the equation,

$$1 - \cos A = 1 - \frac{\cos a - \cos b.\cos c}{\sin b.\sin c}$$

Write $2.\sin^2 \frac{A}{2}$ instead of $1 - \cos A$,

$$2.\sin^2 \frac{A}{2} = \frac{\sin b.\sin c + \cos b.\cos c - \cos a}{\sin b.\sin c}$$

The two terms of the numerator seen above is the function of differences of two angles of cosine,

$$2.\sin^2 \frac{A}{2} = \frac{\cos(b-c) - \cos a}{\sin b.\sin c}$$

Then,

$$\cancel{2}.\sin^2 \frac{A}{2} = \frac{-\cancel{2}.\sin \frac{b-c+a}{2}.\sin \frac{b-c-a}{2}}{\sin b.\sin c}$$

$$\sin^2 \frac{A}{2} = \frac{\sin \frac{(b-c+a)}{2}.\sin \frac{(a+c-b)}{2}}{\sin b.\sin c}$$

Considering $a + b + c = 2.u$

$$\boxed{\begin{aligned} \sin \frac{A}{2} &= \sqrt{\frac{\sin(u-b).\sin(u-c)}{\sin b.\sin c}} \\ \sin \frac{B}{2} &= \sqrt{\frac{\sin(u-a).\sin(u-c)}{\sin a.\sin c}} \end{aligned}}$$

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(u-a).\sin(u-b)}{\sin b.\sin c}}$$

For the cosine functions of half-angle formulas, leave cosA alone at the left side and add 1 to both sides.

$$\boxed{\begin{aligned}\cos\frac{A}{2}&=\sqrt{\frac{\sin u.\sin(u-a)}{\sin b.\sin c}}\\ \cos\frac{B}{2}&=\sqrt{\frac{\sin u.\sin(u-b)}{\sin a.\sin c}}\\ \cos\frac{C}{2}&=\sqrt{\frac{\sin u.\sin(u-c)}{\sin a.\sin b}}\end{aligned}}$$

$$\boxed{\begin{aligned}\tan\frac{A}{2}&=\sqrt{\frac{\sin(u-b).\sin(u-c)}{\sin u.\sin(u-a)}}\\ \tan\frac{B}{2}&=\sqrt{\frac{\sin(u-a).\sin(u-c)}{\sin u.\sin(u-b)}}\\ \tan\frac{C}{2}&=\sqrt{\frac{\sin(u-a).\sin(u-b)}{\sin u.\sin(u-c)}}\end{aligned}}$$

$$\sin A=2.\sin\frac{A}{2}.\cos\frac{A}{2}$$

$$\sin A=2.\sqrt{\frac{\sin(u-b).\sin(u-c)}{\sin b.\sin c}}.\sqrt{\frac{\sin u.\sin(u-a)}{\sin b.\sin c}}$$

$$\sin A=\frac{2}{\sin b.\sin c}.\sqrt{\sin u.\sin(u-a).\sin(u-b).\sin(u-c)}$$

$S=\sqrt{\sin u.\sin(u-a).\sin(u-b).\sin(u-c)}$ ➔Spherical amplitude

$$\sin A=\frac{2.S}{\sin b.\sin c}$$

$$\sin B=\frac{2.S}{\sin a.\sin c}$$

$$\sin C=\frac{2.S}{\sin a.\sin b}$$

Half-Side Formulas:

According to cosine angle theorem;

$$\cos A = -\cos B.\cos C + \sin B.\sin C.\cos a$$

Put cosa to the left side of the equation,

$$\cos a = \frac{\cos B.\cos C + \cos A}{\sin B.\sin C}$$

Add 1 to both sides,

$$1 + \cos a = \frac{\sin B.\sin C + \cos B.\cos C + \cos A}{\sin B.\sin C}$$

Use half angle formula for 1+cosa,

$$2.\cos^2 \frac{a}{2} = \frac{\cos(B-C) + \cos A}{\sin B.\sin C}$$

$$2.\cos^2 \frac{a}{2} = \frac{2.\cos \frac{B-C+A}{2}.\cos \frac{B-C-A}{2}}{\sin B.\sin C}$$

$$A + B + C = 2.v$$

$$\boxed{\begin{aligned} \cos \frac{a}{2} &= \sqrt{\frac{\cos(v-B).\cos(v-C)}{\sin B.\sin C}} \\ \cos \frac{b}{2} &= \sqrt{\frac{\cos(v-A).\cos(v-C)}{\sin A.\sin C}} \\ \cos \frac{c}{2} &= \sqrt{\frac{\cos(v-A).\cos(v-B)}{\sin A.\sin B}} \end{aligned}}$$

If we add 1 to both sides of $\cos a = \dfrac{\cos B.\cos C + \cos A}{\sin B.\sin C}$,

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos v.\cos(v-A)}{\sin B.\sin C}}$$
$$\sin\frac{b}{2} = \sqrt{\frac{-\cos v.\cos(v-B)}{\sin A.\sin C}}$$
$$\sin\frac{c}{2} = \sqrt{\frac{-\cos v.\cos(v-C)}{\sin A.\sin B}}$$

$$\tan\frac{a}{2} = \sqrt{\frac{-\cos v.\cos(v-A)}{\cos(v-B).\cos(v-C)}}$$
$$\tan\frac{b}{2} = \sqrt{\frac{-\cos v.\cos(v-B)}{\cos(v-A).\cos(v-C)}}$$
$$\tan\frac{c}{2} = \sqrt{\frac{-\cos v.\cos(v-C)}{\cos(v-A).\cos(v-B)}}$$

$$\sin a = 2.\sin\frac{a}{2}.\cos\frac{a}{2}$$

Re-arranging the formula as follow,

$$\sin a = \frac{2}{\sin B.\sin C}.\sqrt{-\cos v.\cos(v-A).\cos(v-B).\cos(v-C)}$$

Let T be $T = \sqrt{-\cos v.\cos(v-A).\cos(v-B).\cos(v-C)}$ ➔ called as co-amplitude of spherical triangle

$$\sin a = \frac{2T}{\sin B.\sin C}$$
$$\sin b = \frac{2T}{\sin A.\sin C}$$
$$\sin c = \frac{2T}{\sin A.\sin B}$$

$$2S = \sin A.\sin b.\sin c$$
$$2T = \sin a.\sin B.\sin C$$

$$\frac{S}{T}=\frac{\sin A.\sin b.\sin c}{\sin a.\sin B.\sin C}$$

According to sine theorem,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=M$$

The below can be written,

$$\frac{S}{T}=M$$

Delambre (Molweide) Formulas

$$\sin\frac{A+B}{2}\cos\frac{c}{2}=\cos\frac{C}{2}\cos\frac{a-b}{2}$$

$$\sin\frac{A-B}{2}\sin\frac{c}{2}=\cos\frac{C}{2}\sin\frac{a-b}{2}$$

$$\cos\frac{A+B}{2}\cos\frac{c}{2}=\sin\frac{C}{2}\cos\frac{a+b}{2}$$

$$\cos\frac{A-B}{2}\sin\frac{c}{2}=\sin\frac{C}{2}\sin\frac{a+b}{2}$$

Napier’s Rules

$$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}.\cot\frac{C}{2}$$

$$\tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}.\cot\frac{C}{2}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}.\tan\frac{c}{2}$$

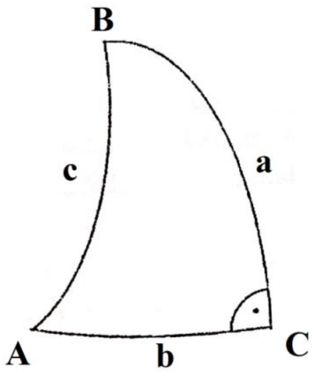
$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}.\tan\frac{c}{2}$$

SPECIAL SPHERICAL TRIANGLES

SOLUTION OF RIGHT-ANGLED TRIANGLES

In every spherical triangle there are six elements, namely, the three sides and the three angles, besides the radius of the sphere, which is supposed constant. The solution of spherical triangles is the process by which, when the values of a sufficient number of the six elements are given, we calculate the values of the remaining elements. It will appear, as we proceed, that when the values of three of the elements are given, those of the remaining three can generally be found. We begin with the right-angled triangle: here two elements, in addition to the right angle, will be supposed known.

Let ABC be a spherical triangle having a right angle at C (90^o); let O be the centre of the sphere. The sum of the spherical triangle is changed between 180^o and 540^o, thus more than one angle may be right angled.
Suppose one of the angles a right angle, as $C=90^o$ for example.



For $C=90^o$;

$\sin C=1$

$\cos C=0$

Sine Theorem:

$$\frac{\sin a}{\sin A}=\frac{\sin c}{\sin C}$$

- $\boxed{\sin a=\sin c.\sin A}$

Similarly;

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

- $\boxed{\sin b = \sin c . \sin B}$

From Cosine-Side theorem:

$$\cos c = \cos a . \cos b + \sin a . \sin b . \cos C$$

- $\boxed{\cos c = \cos a . \cos b}$

This formula is called as **Pythagorean Theorem** rule for right angled triangle. If cosine of an angle theorem is implemented;

$$\boxed{\begin{aligned} &\cos A = -\cos B . \cos C + \sin B . \sin C . \cos a \\ &\cos B = -\cos C . \cos A + \sin C . \sin A . \cos b \\ &\cos C = -\cos A . \cos B + \sin A . \sin B . \cos c \end{aligned}}$$

Then, the following equations may be obtained;

- $\cos A = \sin B . \cos a$
- $\cos B = \sin A . \cos b$
- $\cos c = \cot A . \cot B$

From the cotangent theorem;

$$\sin C . \cot A = \cot a . \sin b - \cos b . \cos C$$

$$\sin C . \cot B = \cot b . \sin a - \cos a . \cos C$$

Then,

- $\sin b = \tan a . \cot A$
- $\sin a = \tan b . \cot B$

From the sine-cosine theorem;

$$\sin a . \cos C = \cos c . \sin b - \sin c . \cos b . \cos A$$

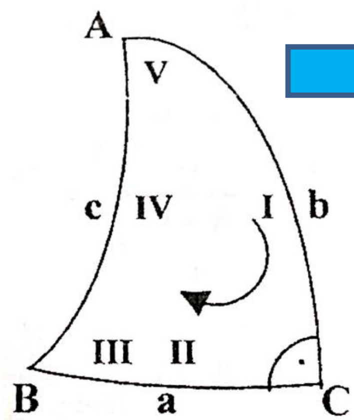
$$\sin b . \cos C = \cos c . \sin a - \sin c . \cos a . \cos B$$

Then,

- $\boxed{\cos A = \cot c . \tan b}$

- $\cos B = \cot c . \tan a$

There are 10 formulas for right angled spherical triangles as given above. So, generalization with two formulas may be seen below. To do this, the elements of right-angled spherical triangle should be numbered in a regular way. Due to a condition that taking the complementary instead of leg;



$$\boxed{\begin{aligned} cosIII &= sinI . sinV \\ cosIII &= cotII . cotIV \end{aligned}}$$

According to the figure:

$$cosB = \sin(90 - b) . sinA$$

$$cosB = \cot(90 - a) . cotc$$

By arranging;

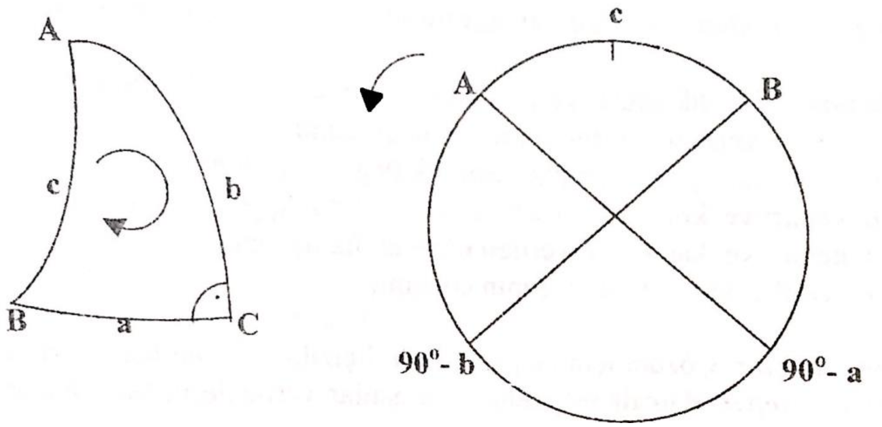
$$\boxed{\begin{aligned} cosB &= cosb . sinA \\ cosB &= tana . cotc \end{aligned}}$$

Using these two formulas for each elements in two ways (clockwise & counter clockwise), ten formulas can be generated.

Napier's Rules

Napier's Rules: The formulas are comprised in two rules, which are called, from their inventor, Napier's Rules of Circular Parts. Napier was also the inventor of Logarithms, and the Rules of Circular Parts were first published by him in a work entitled "*Mirifici Logarithmorum Canonis Descriptio*".

The right angle is left out of consideration; the two sides which include the right angle, the complement of the hypotenuse, and the complements of the other angles are called the circular parts of the triangle.



On the Napier circle, (see right figure above): cosine value of any element is equal to multiplication of the cotangent values of two adjacent elements; and equal to multiplication of the sine values of two opposite elements.

Let the rule be applied to side c:

Having given the two angles A and B;

$$\cos c = \cot A . \cot B$$

Having given the two sides a and b;

$$\cos c = \cos a . \cos b$$

Specifications of Right angled spherical Triangles

- $90^o < A + B < 270^o$
  $-90^o < A - B < 90^o$
- For the leg and the opposite angle; either both of them are acute angles or obtuse angles.
- Hypotenuse is closer to 90^o than other legs.
- If we consider the leg and the opposite angle; angle is closer to 90^o then leg.
- If both legs are obtuse angles or acute angles; hypotenuse is acute angle; if one of the legs is acute angle and the other one is obtuse angle; hypotenuse is obtuse angle.

Solutions of Right-angled spherical Triangles

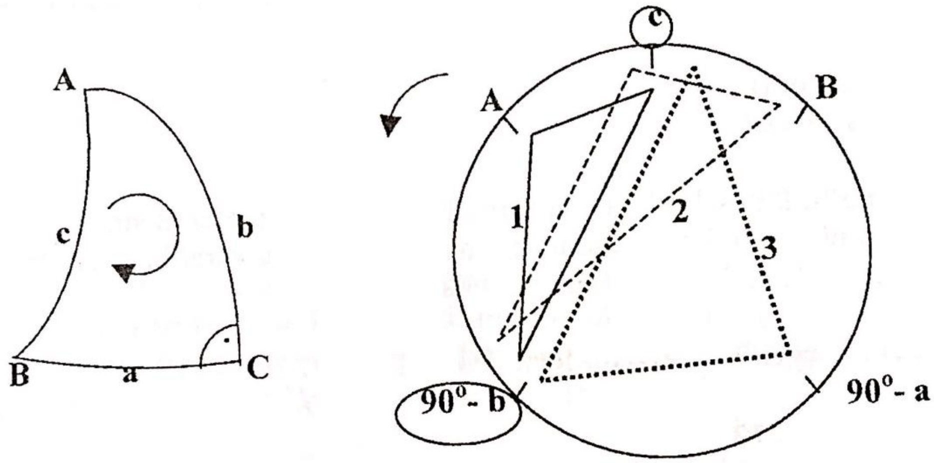
To know 3 elements is enough to solve a spherical triangle. For right angled spherical triangles, one element has right angle, thus it is enough to know 2 elements at all to provide the solution. To solve the right angled spherical triangles, there are 6 cases:

1. Solving the right angled spherical triangles with given hypotenuse and a leg,
2. Solving the right angled spherical triangles with given two legs,
3. Solving the right angled spherical triangles with given hypotenuse and an angle,
4. Solving the right angled spherical triangles with given leg and adjacent angle,

5. Solving the right angled spherical triangles with given leg and opposite angle,
6. Solving the right angled spherical triangles with given two angles,

For solving them, Napier's rule is implemented and Napier circle is drawn.

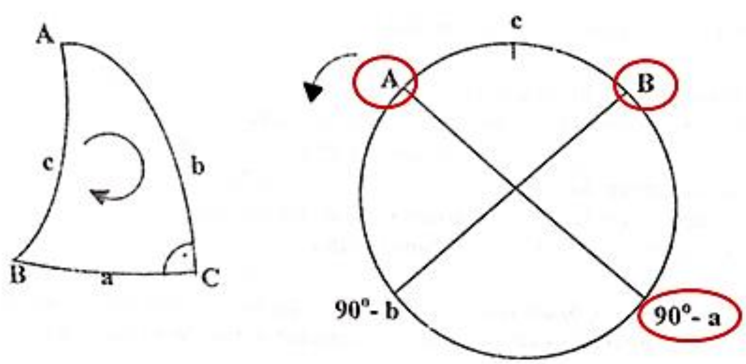
For; C=90°:



1. Solving the right angled spherical triangles with given hypotenuse and a leg,

Given: c,b

To be computed: A, B, a



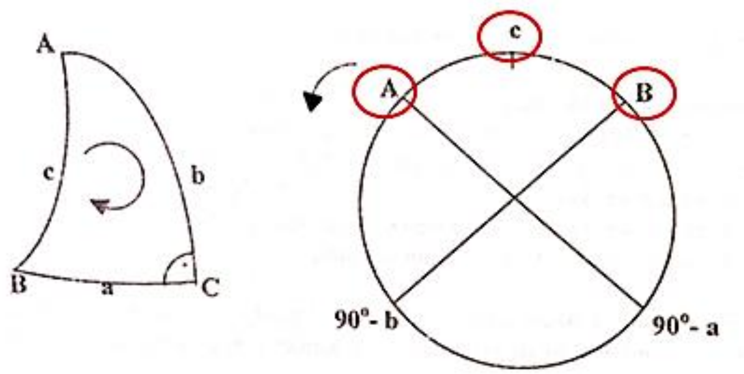
Solution: Implement the Napier's Rule

1- $cosA = cotc.tanb$
2- $sinb = sinc.sinB$ ➔ $sinB = sinb/sinc$
3- $cosc = cosa.cosb$ ➔ $cosa = cosc/cosb$

2. Solving the right angled spherical triangles with given two legs

Given: a,b

To be computed: A, B, c



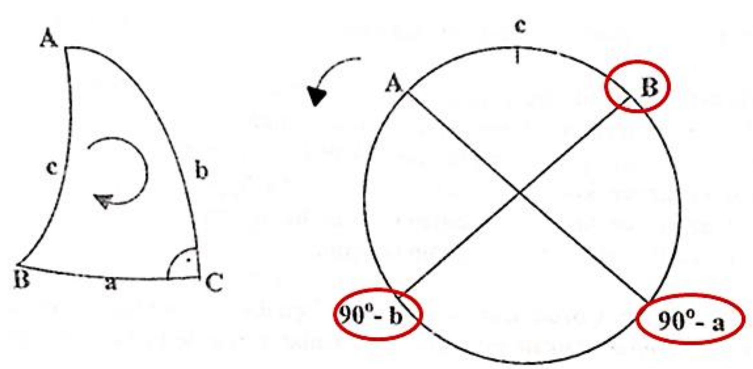
Solution: Implement the Napier's Rule

1- $\cos c = \cos b . \cos a$
2- $\cot B = \sin a . \cot b$
3- $\cot A = \sin b . \cot a$

3. Solving the right angled spherical triangles with given hypotenuse and an angle

Given: c, A

To be computed: B, b, a



Solution: Implement the Napier's Rule

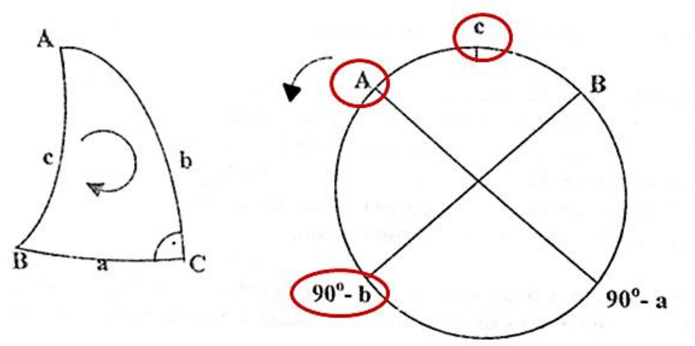
1- $\cot B = \cos c . \tan A$

2- $tanb = cosA.tanc$
3- $sina = sinA.sinc$

4. Solving the right angled spherical triangles with given leg and adjacent angle

Given: a, B

To be computed: A, c, b



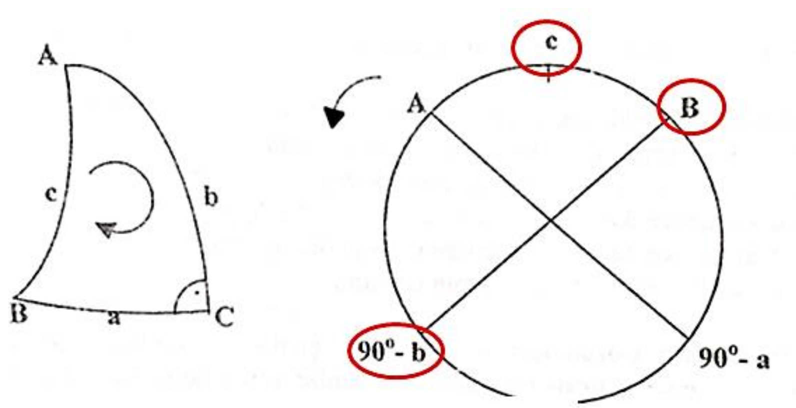
Solution: Implement the Napier's Rule

1- $cotc = cosB.cota$
2- $cosA = sinB.cosa$
3- $tanb = sina.tanB$

5. Solving the right angled spherical triangles with given leg and opposite angle

Given: a, A

To be computed: B, c, b



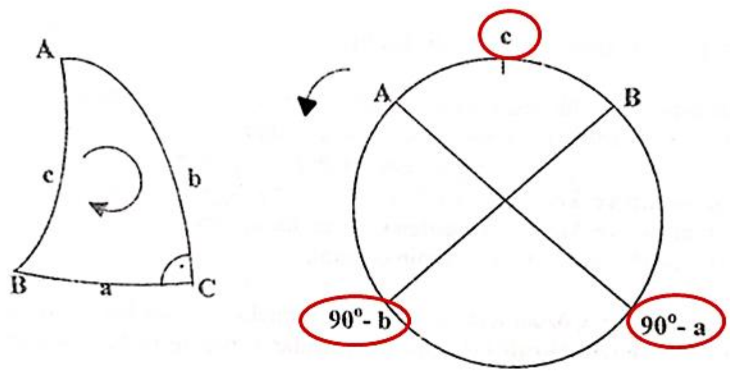
Solution: Implement the Napier's Rule

1- $sinb = cotA.tana$
2- $sinB = cosA/cosa$
3- $sinc = sina/sinA$

6. Solving the right angled spherical triangles with given two angles

Given: A, B

To be computed: a, b, c



Solution: Implement the Napier's Rule

1- $cosa = cosA/sinB$
2- $cosc = cotA.cotB$
3- $cosb = cosB/sinA$

SOLUTION OF SPHERICAL TRIANGLE with A LEG

If one side is 90° in a spherical triangle, it is so called spherical triangle with leg. For example, let c=90° be one leg of spherical triangle, then polar triangle of this may be written as;

$c + C' = 180^o$

$C' = 90^o$ ➔ It will be a right- angled spherical triangle.

If the equations can be written for this polar triangle as per actual spherical triangle elements, the below can be obtained for leg-spherical triangle:

$A' = 180^o - a$

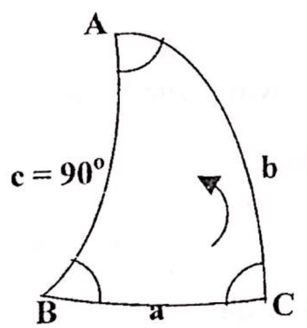
$B' = 180^o - b$

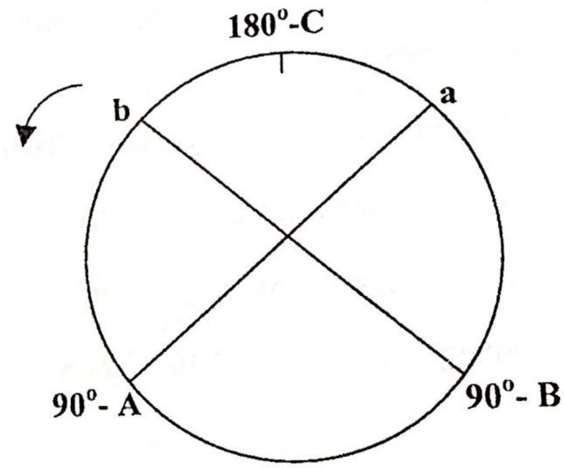
$a' = 180^o - A$

$b' = 180^o - B$

$c' = 180^o - C$

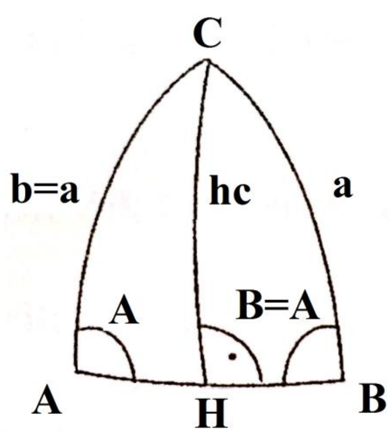
Or, c=90^o can be implemented to the general spherical triangle formulas and the same formulas can be gained as well and with same manner, Napier's rule is implemented. However, the below figure is used for Napier's circle.





SOLUTION OF ISOSCELES and EQUILATERAL SPHERICAL TRIANGLES

If two sides are equal to each other on spherical triangle, it is so called isosceles spherical triangle, e.g. if b=a; thus, then B=A.



SOLUTION OF OBLIQUE SPHERICAL TRIANGLE Involves 6 cases

To solve a spherical triangle, which has totally 6 elements-3 sides and 3 angles, three elements should be known. The solutions as per known elements can be listed as follow into the four cases.

Case 1: 1.a. Two angles and the included side (**ASA**)

Case 2: 1.b. Two angles and a side opposite one of them (**AAS**)

Case 3: 2.a. Two sides and included angle are given (**SAS**)

Case 4: 2.b. Two sides and an angle opposite one of them (**SSA**)

Case 5: 3. Three sides are given (**SSS**)

Case 6: 4. Three angles are given (**AAA**)

1: a. Two angles and a side between these angles (ASA)

Given: A, B, c

To be computed: **C, a, b**

Solution 1:

- According to Cosine of an Angle Theorem
  $\cos C = -\cos A.\cos B + \sin A.\sin B.\cos c$ ➔ Angle C is computed.
- According to Sine Theorem
  $\sin a = \dfrac{\sin c}{\sin C}\sin A$ and $\sin b = \dfrac{\sin c}{\sin C}\sin B$ ➔ sides a and b are computed.

Solution 2:

- According to Cotangent Theorem
  $\cos c.\cos A = \sin c.\cot b - \sin A.\cot B$
- By re-arranging the above formula, leaving cotb alone on one side;
  $\cot b = \dfrac{\cos c.\cos A + \sin A.\cot B}{\sin c}$ ➔ Side b is computed.
- According to Sine Theorem
  $\sin a = \dfrac{\sin b}{\sin B}\sin A$ ➔ Side a is computed.
  $\sin C = \dfrac{\sin B}{\sin b}\sin c$ ➔ Angle C is computed.

Solution 3:

- According to Napier's Rule

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}.\tan\frac{c}{2}$$

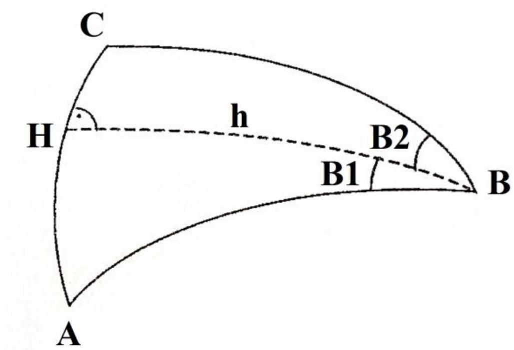
⇨ Sides a and b are computed.

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}.\tan\frac{c}{2}$$

- According to Sine Theorem

$$\sin C=\frac{\sin A}{\sin a}\sin c=\frac{\sin B}{\sin b}\sin c$$ ➔Angle C is computed.

Solution 4: Spherical triangle can be divided into right angled triangles for solution. For ABC spherical triangle, from B, a perpendicular line is drawn to AC.



- According to AHB triangle:

$$\tan AH=\cos A.\tan c$$
$$\sin h=\sin A.\sin c$$
$$\cot B_1=\tan A.\cos c$$
$$B_2=B-B_1$$

- According to CHB triangle:

$$\cot a=\cos B_2.\coth$$
$$\cos C=\sin B_2.\cosh$$
$$\tan CH=\sinh.\tan B_2$$
$$b=AH+CH$$

1: b. Two angles and a side opposite to one of these angles (AAS)

Given: A, B, b

<u>To be computed:</u> C, a, c

Solution 1:

- According to Sine Theorem

  $\sin a = \dfrac{\sin b}{\sin B}\sin A$ ➔ Side a is computed.

- According to Napier's Rule

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}.\tan\frac{c}{2} \Rightarrow \tan\frac{c}{2} = \tan\frac{a+b}{2}.\frac{\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}}$$

➔ Side c and Angle C are computed.

$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}.\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = \tan\frac{A+B}{2}.\frac{\cos\frac{a+b}{2}}{\cos\frac{a-b}{2}}$$

Solution 2:

- According to Cotangent Theorem

  $\cos A.\cos c = \cot b.\sin c - \sin A.\cot B$

- Multiple both sides by "$\tan b$"

  $\sin c - \cos A.\tan b.\cos c - \sin A.\cot B.\tan b = 0$

- Here, to determine c, trigonometric equation should be solved.
- Transformation of $\tan\varphi = \cos A.\tan b$ is implemented to the equation

  $\sin c - \tan\varphi.\cos c - \sin A.\cot B.\tan b = 0$

- Write $\sin\varphi / \cos\varphi$ Multiple both sides by "$\tan\varphi$"

  $\sin c - (\sin\varphi / \cos\varphi).\cos c - \sin A.\cot B.\tan b = 0$

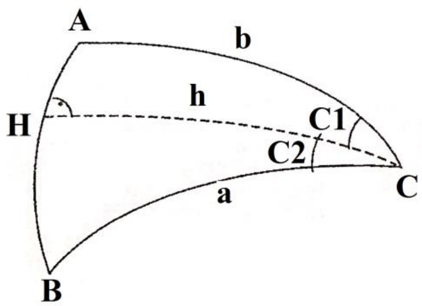
- Multiple both sides by "$\cos\varphi$"

  $\sin c.\cos\varphi - \cos c.\sin\varphi - \sin A.\cot B.\tan b.\cos\varphi = 0$

  $\sin(c-\varphi) = \sin A.\cot B.\tan b.\cos\varphi$ ➔ Side c is computed.

- Continue to the solution with sine theorem

Solution 3: Spherical triangle can be divided into right angled triangles for solution.



- From AHC right angled spherical triangle:
  $\sin h = \sin A . \sin b$
  $\tan AH = \cos A . \tan b$
  $\cot C_1 = \cos b . \tan A$
- From BHC right angled spherical triangle:
  $\sin a = \sin h / \sin B$
  $\sin BH = \cot B . \tan h$
  $c = AH + BH$
  $\sin C_2 = \cos B / \cos h$

2: a. Two sides and an angle between these sides (SAS)

Given: a, b, C

To be computed: c, A, B

Solution 1:

- According to Cosine of a side Theorem
- $\cos c = \cos a . \cos b + \sin a . \sin b . \cos C$ ➔ Side c is computed.
- According to Sine Theorem

  $$\sin A = \frac{\sin C}{\sin c} \sin a$$ ➔ Angle A is computed.

  $$\sin B = \frac{\sin C}{\sin c} \sin b$$ ➔ Angle B is computed.
- For side c, there is unique solution. However, for Angles B and C, due to using sine function, the operation will give two solutions. So, we should find out unique values of them by using below inequations:
- These comparisons should be done:
  If $a>c$, or $a<c$, then $A>C$ or $A<C$ and if $b>c$, or $b<c$, then $B>C$ or $B<C$

Solution 2:

- According to Cotangent Theorem

  $$\cot B = \frac{\sin a \cot b - \cos C.\cos a}{\sin C}$$ ➔ Angle B computed.

  $$\cot A = \frac{\sin b \cot a - \cos C.\cos b}{\sin C}$$ ➔ Angle A computed.

- According to Sine Theorem

  $$\sin c = \frac{\sin b}{\sin B}\sin C = \frac{\sin a}{\sin A}\sin C$$ ➔ Side c computed.

- Here, For A and B, there is unique solution. However, for c, due to using sine function, the operation will give two solutions. So, we should find out unique values of them by using below inequations:
- These comparisons should be done:
  If C>A, or C<A, then c>a or c<a and if C>B, or C<B, then c>b or c<b

Solution 3:

- According to Napier's Rule

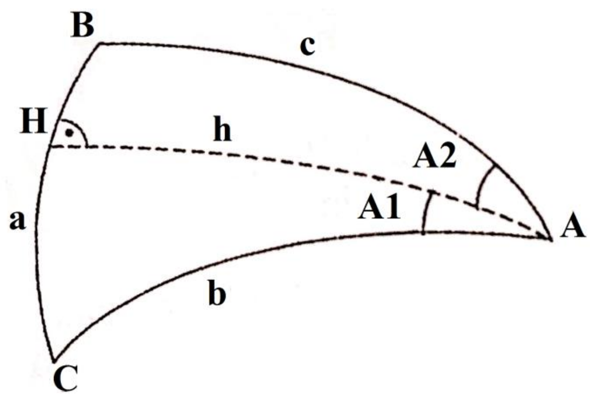
  $$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}.\cot\frac{C}{2}$$

  $$\tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}.\cot\frac{C}{2}$$

  $$\tan\frac{c}{2} = \tan\frac{a+b}{2}.\frac{\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}} = \tan\frac{a-b}{2}.\frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}$$

- Angles "A" and "B", and side c are computed.

Solution 4: Spherical triangle can be divided into right angled triangles for solution.



- From AHC right angled spherical triangle:
  $\tan CH = \cos C . \tan b$
  $\cot A_1 = \cos b . \tan C$
  $BH = a - CH$
- From AHB right angled spherical triangle:
  $\cos c = \cos BH . \cos h$
  $\cot B = \sin BH . \cot h$
  $\cot A_2 = \sin h . \cot BH$
  $A = A_1 + A_2$

2: b. Two sides and an angle opposite to one of these sides (SSA)

Given: a, b, B

To be computed: A, C, c

Solution 1:

- According to Sine Theorem and Napier's equations

$$\sin A = \frac{\sin B}{\sin b} \sin a$$ ➔ Angle A is computed.

$$\tan \frac{c}{2} = \tan \frac{a+b}{2} . \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}}$$ ➔ Side c is computed.

$$\cot \frac{C}{2} = \tan \frac{A+B}{2} . \frac{\cos \frac{a+b}{2}}{\cos \frac{a-b}{2}}$$ ➔ Angle C is computed.

Solution 2:

- According to Cotangent Theorem (Trigonometric equation solution)
  $\cos C.\cos a = \cot b.\sin a - \sin C.\cot B$
- Multiple both sides by "$\tan B$"
  $\sin C + \cos a.\tan B.\cos C = \sin a.\cot b.\tan B$
- Here, to determine c, trigonometric equation should be solved.
- Transformation of $\tan\varphi = \cos a.\tan B$ is implemented to the equation
  $\sin C + \tan\varphi.\cos C = \sin a.\cot b.\tan B$
- Write $\sin\varphi / \cos\varphi$ Multiple both sides by "$\tan\varphi$"
  $\sin C + (\sin\varphi / \cos\varphi).\cos C = \sin a.\cot b.\tan B$
- Multiple both sides by "$\cos\varphi$"
  $\sin C.\cos\varphi + \cos C.\sin\varphi = \sin a.\cot b.\tan B.\cos\varphi$
  $\sin(C + \varphi) = \sin a.\cot b.\tan B.\cos\varphi$ ➔ C is computed.
- Continue to the solution with sine theorem

Solution 3: Spherical triangle can be divided into right angled triangles for solution.

3: Three sides (SSS)

Given: a, b, c

To be computed: **A, B, C**

Solution 1:

- According to Cosine of a side Theorem

$$\cos A = \frac{\cos a - \cos b.\cos c}{\sin b.\sin c}$$

$$\cos B = \frac{\cos b - \cos a.\cos c}{\sin a.\sin c}$$

$$\cos C = \frac{\cos c - \cos a.\cos b}{\sin a.\sin b}$$

Solution 2:

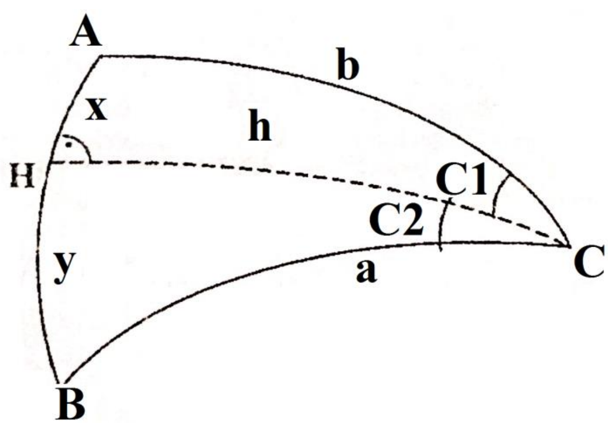
- According to Half-Angle Formulas

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(u-b).\sin(u-c)}{\sin u.\sin(u-a)}}$$

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(u-a).\sin(u-c)}{\sin u.\sin(u-b)}}$$

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(u-a).\sin(u-b)}{\sin u.\sin(u-c)}}$$

Solution 3: Spherical triangle can be divided into right angled triangles for solution.



- From CHA and CHB right angled spherical triangles:
  $\cos b=\cos x.\cos h$
  $\cos a=\cos y.\cos h$
- Proportion the above two equations
  $$\frac{\cos x}{\cos y}=\frac{\cos b}{\cos a}$$
- And considering $c=x+y$
  $$\frac{\cos x-\cos y}{\cos x+\cos y}=\frac{\cos b-\cos a}{\cos b+\cos a}$$
- According to transformation equations
  $$\frac{-2.\sin\frac{x+y}{2}.\sin\frac{x-y}{2}}{2.\cos\frac{x+y}{2}.\cos\frac{x-y}{2}}=\frac{-2.\sin\frac{b+a}{2}.\sin\frac{b-a}{2}}{2.\cos\frac{b+a}{2}.\cos\frac{b-a}{2}}$$
- Arranging the equation, then
  $$\tan\frac{x+y}{2}.\tan\frac{x-y}{2}=\tan\frac{b+a}{2}.\tan\frac{b-a}{2}$$

$$\tan\frac{x-y}{2}=\tan\frac{b+a}{2}.\tan\frac{b-a}{2}.\cot\frac{c}{2}$$ ➔"x" and "y" are computed.

- From CHA right angled spherical triangle:
  $\cos A=\cot b.\tan x$ ➔ A is computed.
  $\sin C_1=\sin x/\sin b$ ➔ C_1 is computed.
- From CBH right angled spherical triangle:
  $\cos B=\cot a.\tan y$ ➔ B is computed.
  $\sin C_2=\sin y/\sin a$ ➔ C_2 is computed.
  $C=C_1+C_2$

4: Three angles (AAA)

Given: A, B, C

To be computed: **a, b, c**

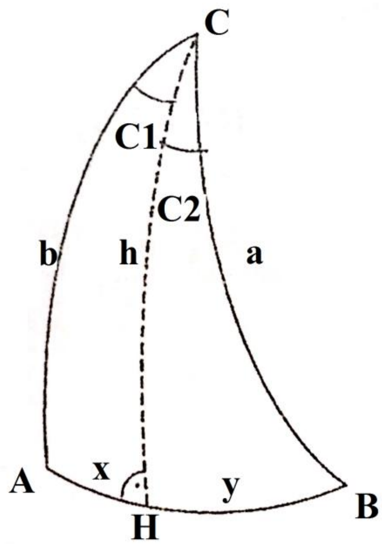
Solution 1:

- According to Cosine of an angle Theorem
- $\cos a=\dfrac{\cos A+\cos B.\cos C}{\sin B.\sin C}$
- $\cos b=\dfrac{\cos B+\cos A.\cos C}{\sin A.\sin C}$
- $\cos c=\dfrac{\cos C+\cos A.\cos B}{\sin A.\sin B}$

Solution 2:

- According to Half-side formula
- $\cot\dfrac{a}{2}=\sqrt{\dfrac{\cos(v-B)\cos(v-C)}{-\cos v.\cos(v-A)}}$
- $\cot\dfrac{b}{2}=\sqrt{\dfrac{\cos(v-A)\cos(v-C)}{-\cos v.\cos(v-B)}}$
- $\cot\dfrac{c}{2}=\sqrt{\dfrac{\cos(v-A)\cos(v-B)}{-\cos v.\cos(v-C)}}$

Solution 3: Spherical triangle can be divided into right angled triangles for solution.



- From CHA and CHB right angled spherical triangles:

$$\cos A = \cos h . \sin C_1$$
$$\cos B = \cos h . \sin C_2$$

- Proportion the above two equations

$$\frac{\sin C_1}{\sin C_2} = \frac{\cos A}{\cos B}$$

- And considering $C = C_1 + C_2$

$$\frac{\sin C_1 - \sin C_2}{\sin C_1 + \sin C_2} = \frac{\cos A - \cos B}{\cos A + \cos B}$$

- According to transformation equations

$$\frac{2.\sin\frac{C_1 - C_2}{2}.\cos\frac{C_1 + C_2}{2}}{2.\sin\frac{C_1 + C_2}{2}.\cos\frac{C_1 - C_2}{2}} = \frac{-2.\sin\frac{A+B}{2}.\sin\frac{A-B}{2}}{2.\cos\frac{A+B}{2}.\cos\frac{A-B}{2}}$$

- Arranging the equation, then

$$\tan\frac{C_1 - C_2}{2}.\cot\frac{C_1 + C_2}{2} = -\tan\frac{A+B}{2}.\tan\frac{A-B}{2}$$

$$\tan\frac{C_1 - C_2}{2} = \tan\frac{A+B}{2}.\tan\frac{B-A}{2}.\tan\frac{C}{2}$$ ➔" C_1 " and " C_2 " are computed.

- From CHA right angled spherical triangle:

$\cos b = \cot C_1 . \cot A$ ➔ b is computed.

$\cos x = \cos C_1 / \sin A$ ➔ x is computed.

- From CBH right angled spherical triangle:

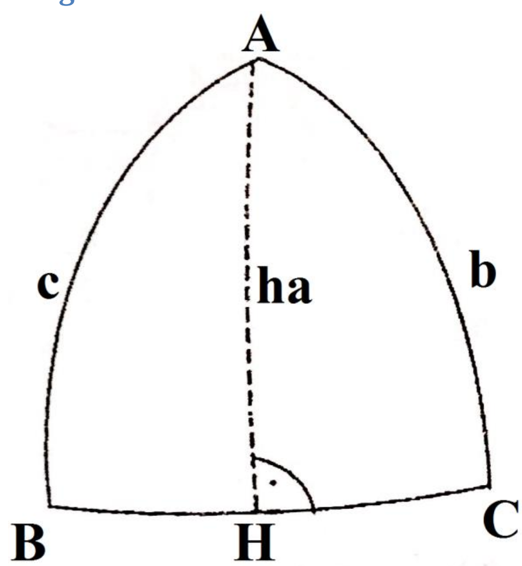
  $\cos a = \cot C_2 . \cot B$ ➔ a is computed.

  $\cos y = \cos C_2 / \sin B$ ➔ y is computed.

  $c = x + y$

Solutions of the Other Elements on Spherical Triangle

Height of the Spherical Triangle



$\sinh_a = \sin c . \sin B = \sin b . \sin C$

$\sinh_b = \sin a . \sin C = \sin c . \sin A$

$\sinh_c = \sin a . \sin B = \sin b . \sin A$

$$\sin A = \frac{2.S}{\sin b . \sin c}$$

$$\sin B = \frac{2.S}{\sin a . \sin c}$$

$$\sin C = \frac{2.S}{\sin a . \sin b}$$

$$\sin a = \frac{2.T}{\sin B . \sin C}$$

$$\sin b = \frac{2.T}{\sin A . \sin C}$$

$$\sin c = \frac{2.T}{\sin A . \sin B}$$

$$\sinh_a = \frac{2S}{\sin a} = \frac{2T}{\sin A}$$

$$\sinh_b = \frac{2S}{\sin b} = \frac{2T}{\sin B}$$

$$\sinh_c = \frac{2S}{\sin c} = \frac{2T}{\sin C}$$

From the 1st and 3rd equations, unique values for each element can be computed. For the 2nd equation, there are two solutions. Even either B and b should be acute angles or obtuse angles at the same time, unique solution can then be obtained.

If A is close to “0” or “180”, It can be computed as follow;

$cotA = cosc.tanB$ or $cotA = sinb.cota$

If any side is close to “0” or “180”, It can be computed as follow;

$\tan a = cosB.tanc$ or $tana = sinb.tanA$

If b, c or B is close to “0” or “180”, It can be computed as follow;

References:

Prof. Dr. Yuji Murayama Surantha Dassanayake, "Fundamentals of Surveying Theory and Samples Exercises", Division of Spatial Information Science Graduate School Life and Environment Sciences University of Tsukuba,

available at: http://giswin.geo.tsukuba.ac.jp/sis/tutorial/fundamentals_of_surveying.pdf

URL: http://www.ce.memphis.edu/1112/notes/project_3/traverse/Surveying_angles.pdf