Heat and Mass Transfer, 3rd Edition
Yunus A. Cengel
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CHAPTER 2
HEAT CONDUCTİON EQUATİON

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OUTLİNE

- Introduction
- One-dimensional heat conduction equation
- General heat conduction equation
- Boundary and initial conditions
- Solution of one-dimensional heat conduction problems
- Heat generation in a solid
- Variable thermal conductivity
- Conclusions

OBJECTİVES

- To understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional.
- To obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case.
- To identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions.
- To solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux.
- To analyze one-dimensional heat conduction in solids that involve heat generation.
- To evaluate heat conduction in solids with temperature-dependent thermal conductivity.

INTRODUCTİON

Although heat transfer and temperature are closely related, they are of a different nature.

Temperature has only magnitude

it is a *scalar* quantity.

Heat transfer has direction as well as magnitude

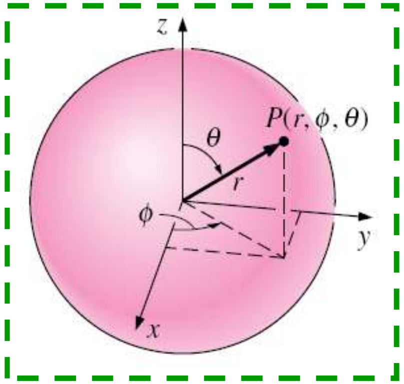
it is a *vector* quantity.

We work with a coordinate system and indicate direction with plus or minus signs.

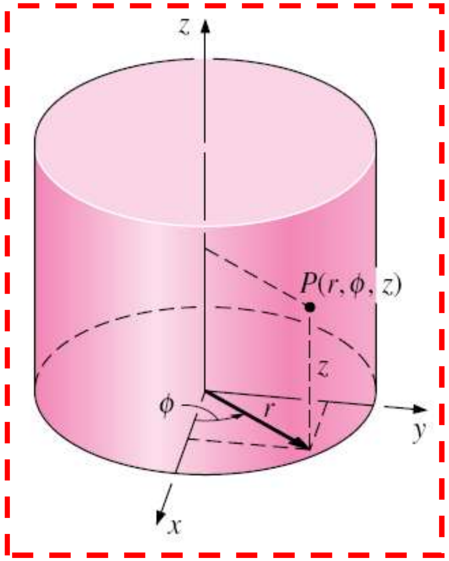
The driving force for any form of heat transfer is the *temperature difference.*

- **The larger the temperature difference,**
- **The larger the rate of heat transfer.**

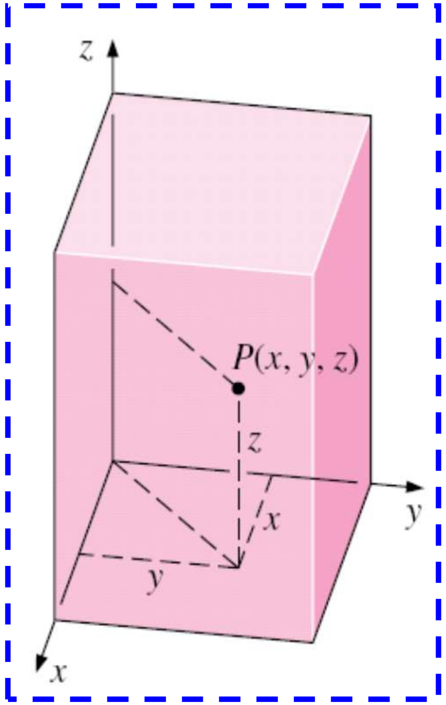
spherical



cylindrical



rectangular



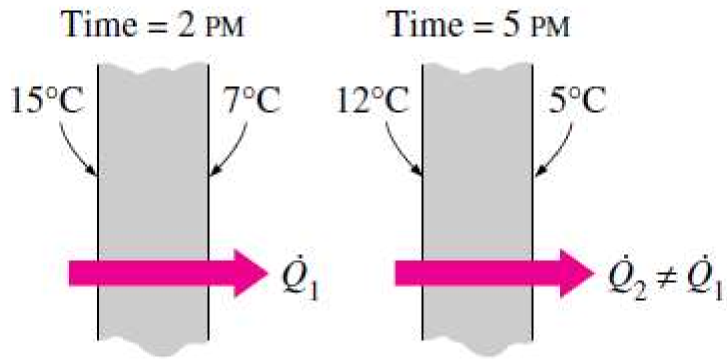
Classification of conduction heat transfer problems:

- Steady versus transient heat transfer,
- Multidimensional heat transfer,
- Heat generation.

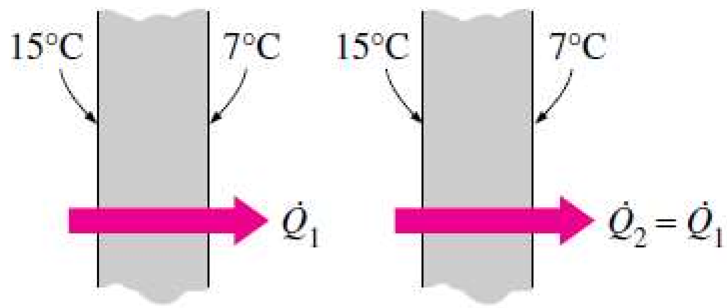
Steady versus transient heat transfer

- ***Transient*** implies *variation with time* or *time dependence*

- ***Steady*** implies *no change* with time at any point within the medium



(*a*) Transient



(*b*) Steady

Multidimensional heat transfer

Heat transfer problems are also classified as being:

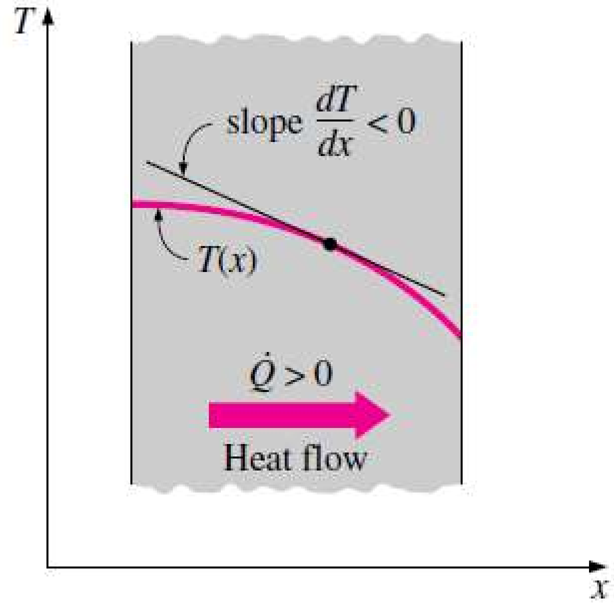
- *One-dimensional*,
- *Two dimensional*,
- *Three-dimensional.*

In the most general case, heat transfer through a medium is 3-D. However, some problems can be classified as 2- or 1-D depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.

The rate of heat conduction through a medium in a specified direction is expressed by **Fourier's law of heat conduction** for 1-D heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

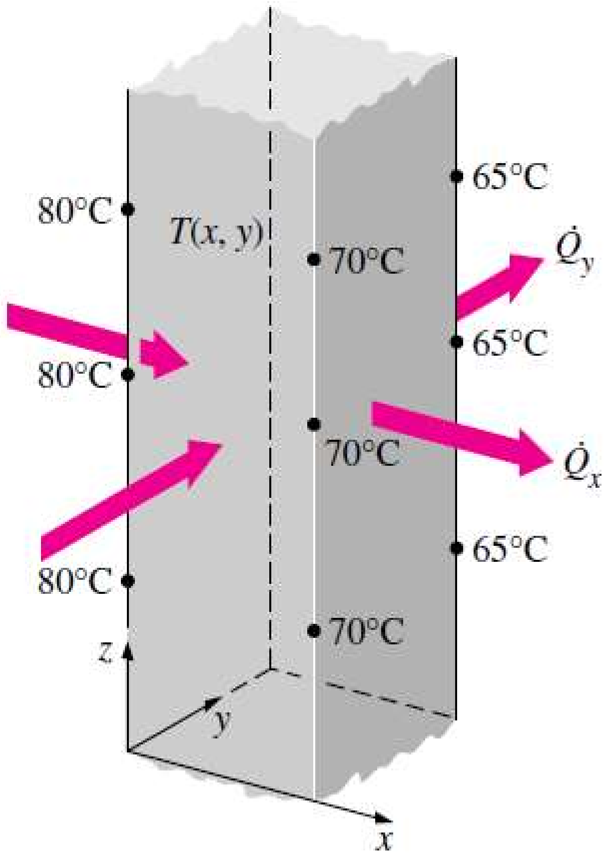
Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive x -direction.



Multidimensional heat transfer

1-D if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.

2-D if the temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction is negligible.



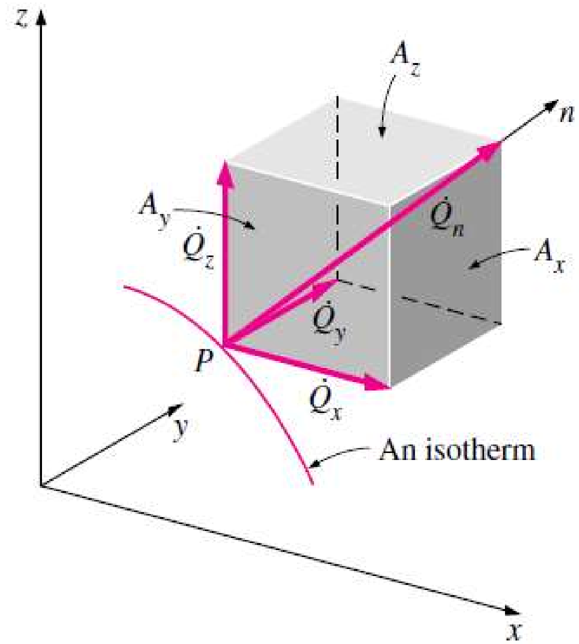
Two-dimensional heat transfer in a long rectangular bar.

Fourier's Law of Heat Conduction

The heat flux vector at a point *P* on the surface must be perpendicular to the surface, and it must point in the direction of decreasing temperature

If *n* is the normal of the isothermal surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{\partial T}{\partial n} \qquad \text{(W)}$$



In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k}$$

which can be determined from Fourier's law as

$$\dot{Q}_x = -kA_x\frac{\partial T}{\partial x}, \qquad \dot{Q}_y = -kA_y\frac{\partial T}{\partial y}, \qquad \text{and} \qquad \dot{Q}_z = -kA_z\frac{\partial T}{\partial z}$$

HEAT GENERATİON

Examples:

- Electrical energy being converted to heat at a rate of I^2R,
- Fuel elements of nuclear reactors,
- Exothermic chemical reactions.

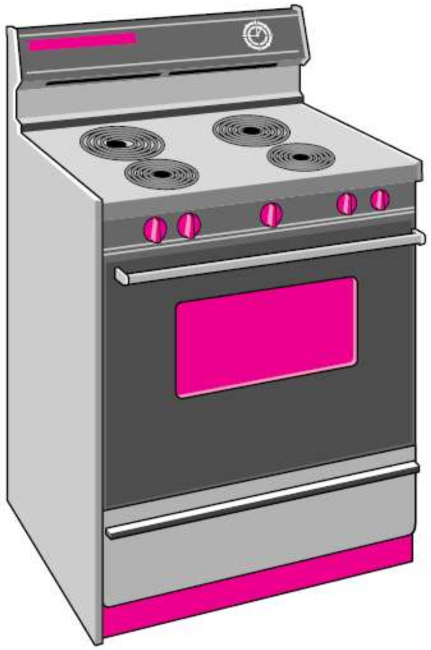
Heat generation is a *volumetric phenomenon.*

The rate of heat generation units : W/m³ or Btu/h · ft³.

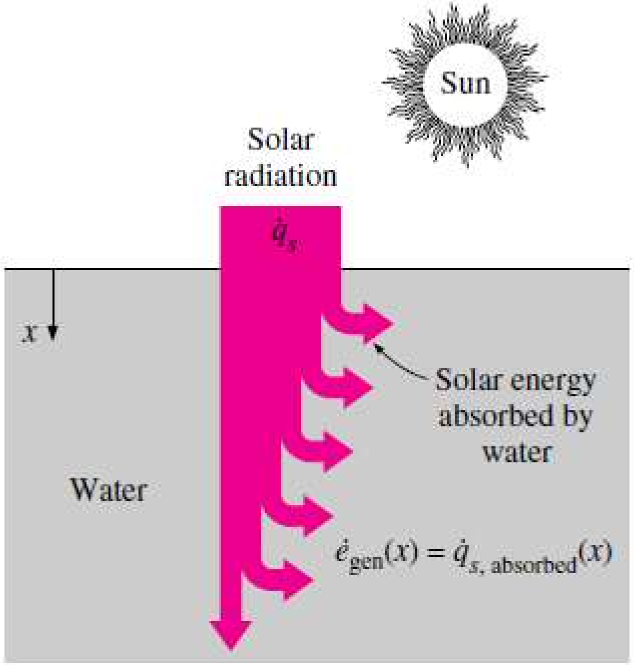
The rate of heat generation in a medium may vary with time as well as position within the medium.

The *total* rate of heat generation in a medium of volume *V* :

$$\boxed{\dot{E}_{\text{gen}} = \int_V \dot{e}_{\text{gen}}\, dV \qquad \text{(W)}}$$



Heat is generated in the heating coils of an electric range as a result of the conversion of electrical energy to heat.

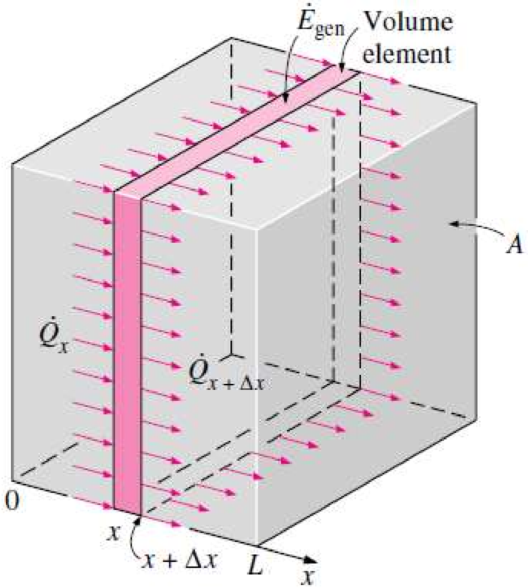


The absorption of solar radiation by water can be treated as heat generation.

1-D HEAT CONDUCTİON EQUATİON - PLANE WALL

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



$A_x = A_{x+\Delta x} = A$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta x \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta x \end{cases}$$

Substituting into above equation, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta x$, taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\boxed{\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}}$$

The area *A* is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The one-dimensional conduction equation may be reduces to the following forms under special conditions

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{d^2T}{dx^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}} = 0)$

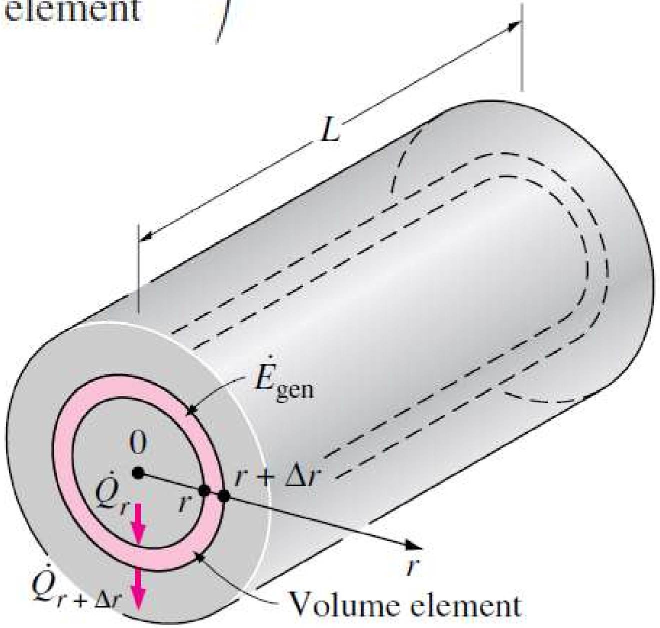
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{e}_{\text{gen}} = 0)$

$$\frac{d^2T}{dx^2} = 0$$

1-D Heat Conduction Equation - Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$



$$\boxed{\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t}}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta r \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r \end{cases}$$

Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta r$, taking the limit as $\Delta r \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Noting that the area varies with the independent variable r according to $A=2\pi rL$, the one dimensional transient heat conduction equation in a long cylinder becomes

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{\text{gen}}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{\text{gen}}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

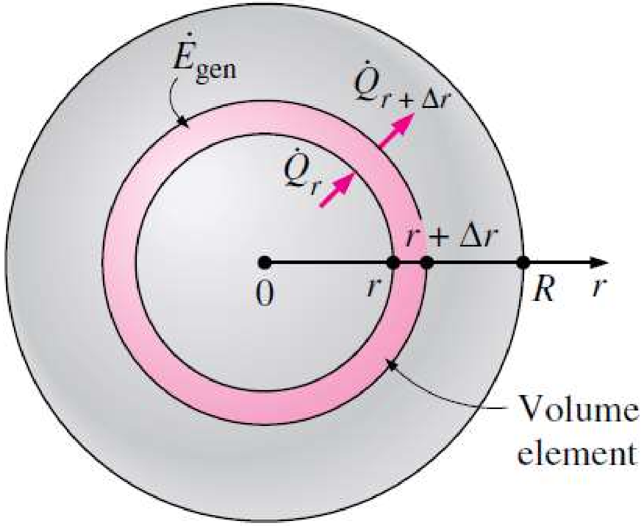
The one-dimensional conduction equation may be reduces to the following forms under special conditions

(1) *Steady-state:* $(\partial/\partial t=0)$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{\text{gen}}}{k}=0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}}=0)$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t=0 \text{ and } \dot{e}_{\text{gen}}=0)$ $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

1-D Heat Conduction Equation - Sphere



Variable conductivity

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* ($\partial/\partial t = 0$)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

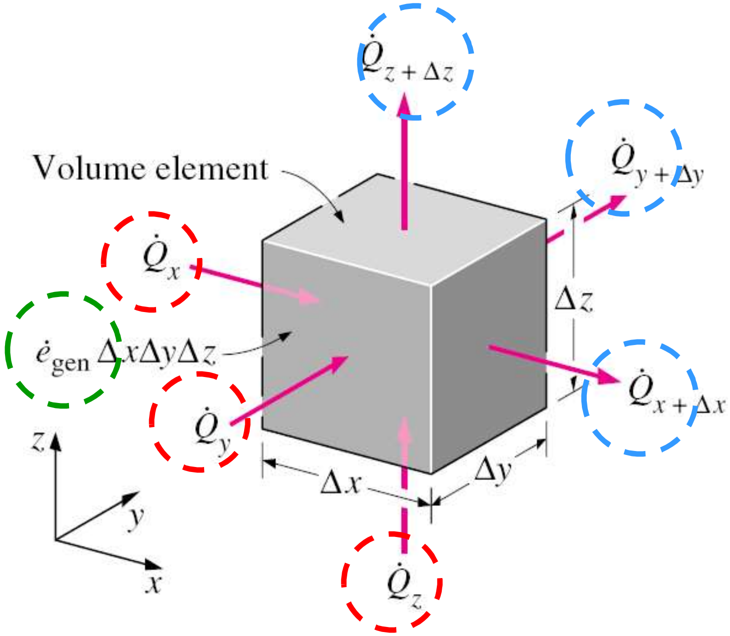
(2) *Transient, no heat generation:* ($\dot{e}_{\text{gen}} = 0$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* ($\partial/\partial t = 0$ and $\dot{e}_{\text{gen}} = 0$)

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad \text{or} \qquad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

GENERAL HEAT CONDUCTİON EQUATİON



Rate of heat conduction at *x*, *y*, and *z* **-** **Rate of heat conduction at *x+Δx*, *y+Δy*, *and z+Δz*** **+** **Rate of heat generation inside the element** **=** **Rate of change of the energy content of the element**

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Constant conductivity:

$$\underbrace{\overbrace{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}}^{\text{Two-dimensional}}+\frac{\partial^2 T}{\partial z^2}}_{\text{Three-dimensional}}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

called **Fourier-Biot Equation**

1) Steady-state:

called **Poisson Equation**

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

called **diffusion Equation**

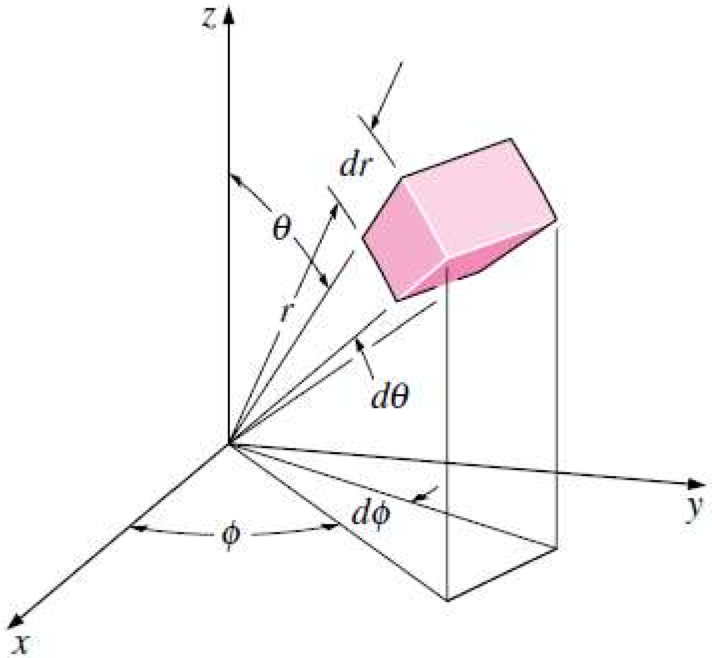
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation:

called **Lapplace Equation**

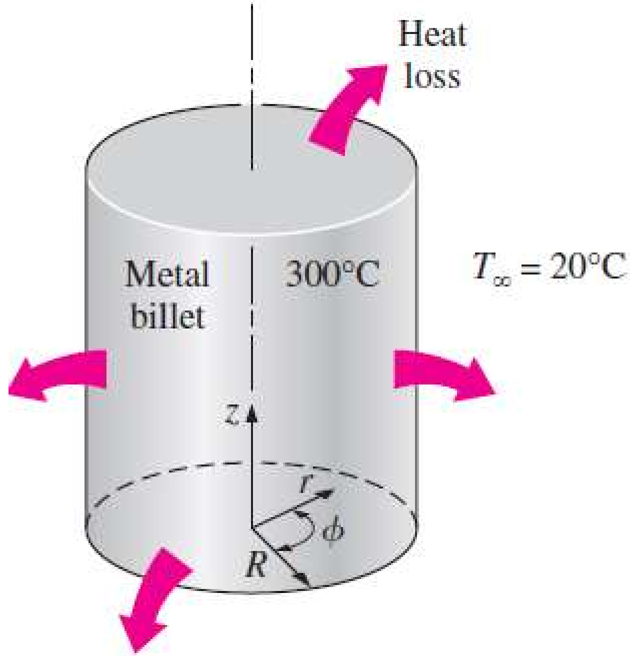
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$

Cylindrical coordinates



$$x = r\cos\phi,$$
$$y = r\sin\phi,$$
$$z = z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$



Spherical coordinates

$$x = r\cos\phi\sin\theta,$$
$$y = r\sin\phi\sin\theta,$$
$$z = \cos\theta$$

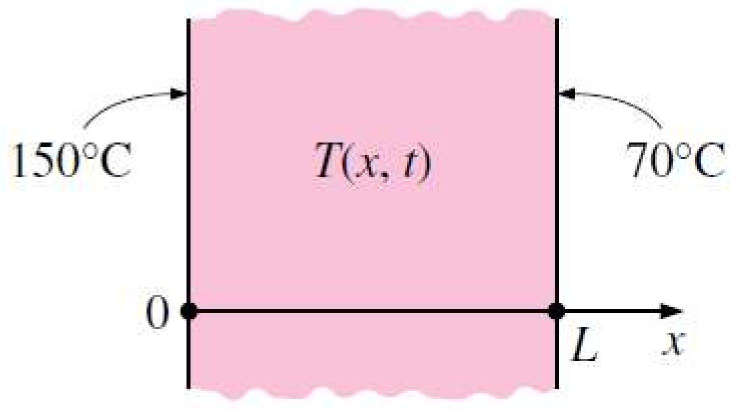
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

BOUNDARY AND INİTİAL CONDİTİONS

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Specified temperature boundary condition

For one-dimensional heat transfer through a plane wall of thickness *L*, for example, the specified temperature boundary conditions can be expressed as



$$T(0, t) = 150°\text{C}$$
$$T(L, t) = 70°\text{C}$$

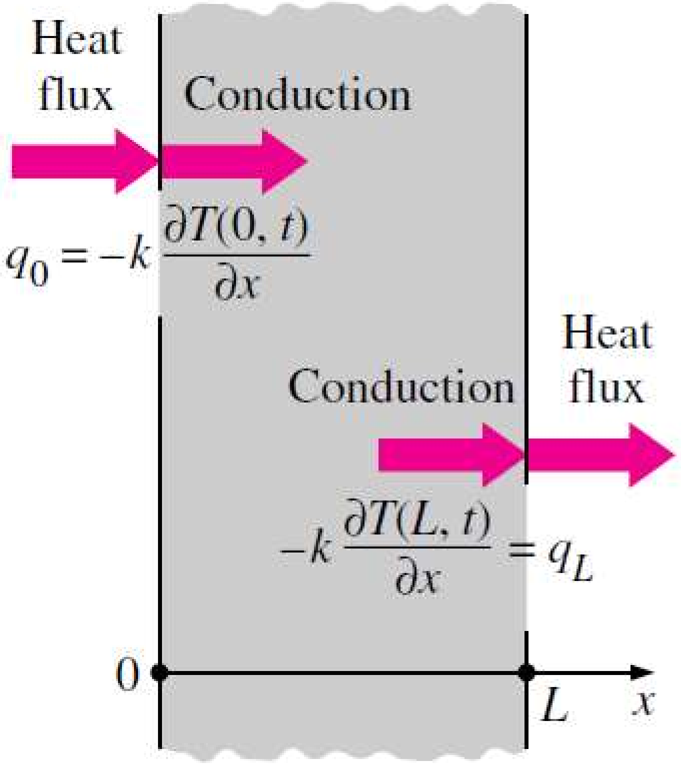
$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as

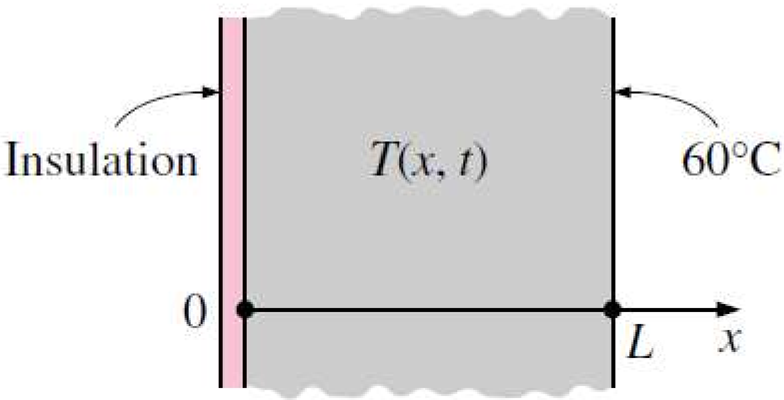
$$\dot{q} = -k\frac{dT}{dx} = \left(\text{Heat flux in the positive x-direction} \right)$$



The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

TWO SPECİAL CASES

Insulated boundary

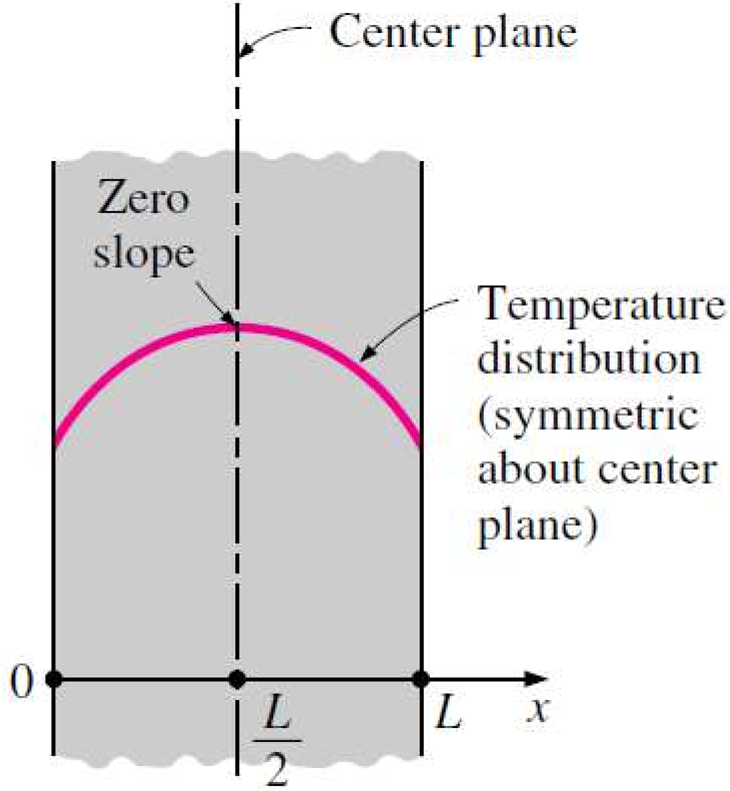


$$\frac{\partial T(0,t)}{\partial x} = 0$$

$$T(L,t) = 60°\text{C}$$

$$k\frac{\partial T(0,t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0,t)}{\partial x} = 0$$

Thermal symmetry



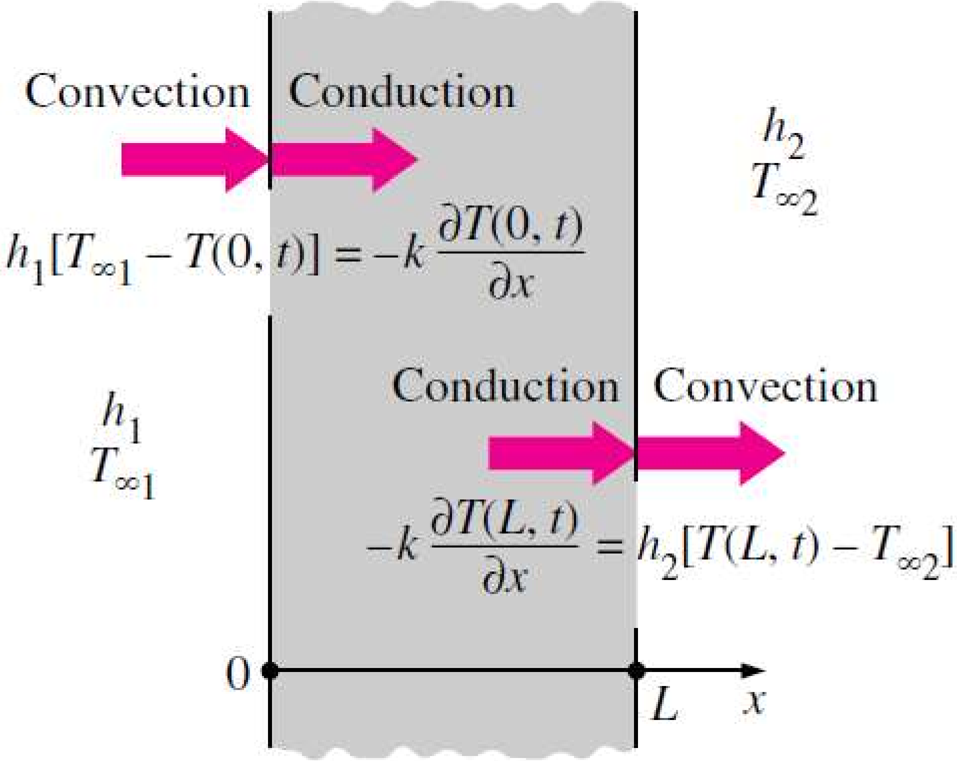
$$\frac{\partial T(L/2,t)}{\partial x} = 0$$

Convection boundary condition

$$\left(\begin{array}{c}\text{Heat conduction}\\ \text{at the surface in}\\ \text{a selected}\\ \text{direction}\end{array}\right) = \left(\begin{array}{c}\text{Heat convection}\\ \text{at the surface in}\\ \text{the same}\\ \text{direction}\end{array}\right)$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$

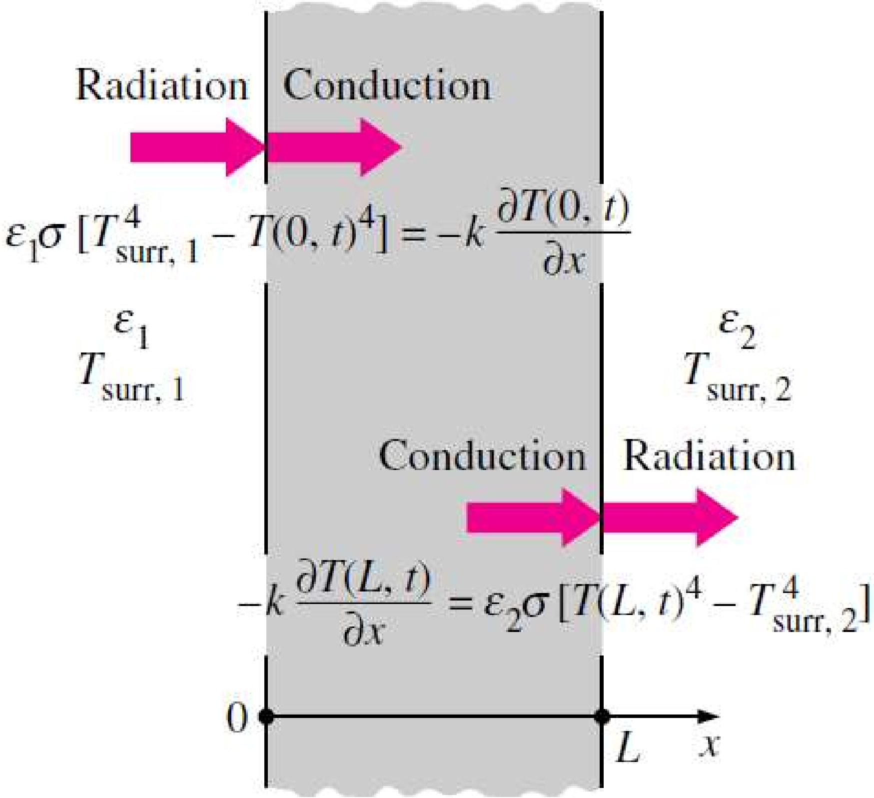


Radiation boundary condition

$$\begin{pmatrix}\text{Heat conduction} \\ \text{at the surface in a} \\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Radiation} \\ \text{exchange at the} \\ \text{surface in} \\ \text{the same direction}\end{pmatrix}$$

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma[T^4_{\text{surr},1} - T(0,t)^4]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\sigma[T(L,t)^4 - T^4_{\text{surr},2}]$$

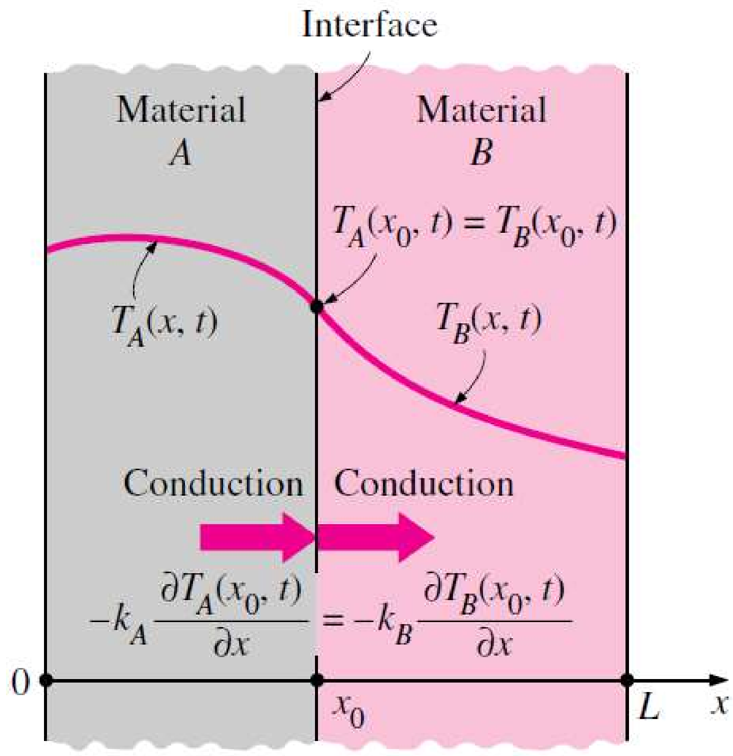


Interface boundary conditions

At the interface the requirements are:
(1) two bodies in contact must have the *same temperature* at the area of contact,
(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same*.

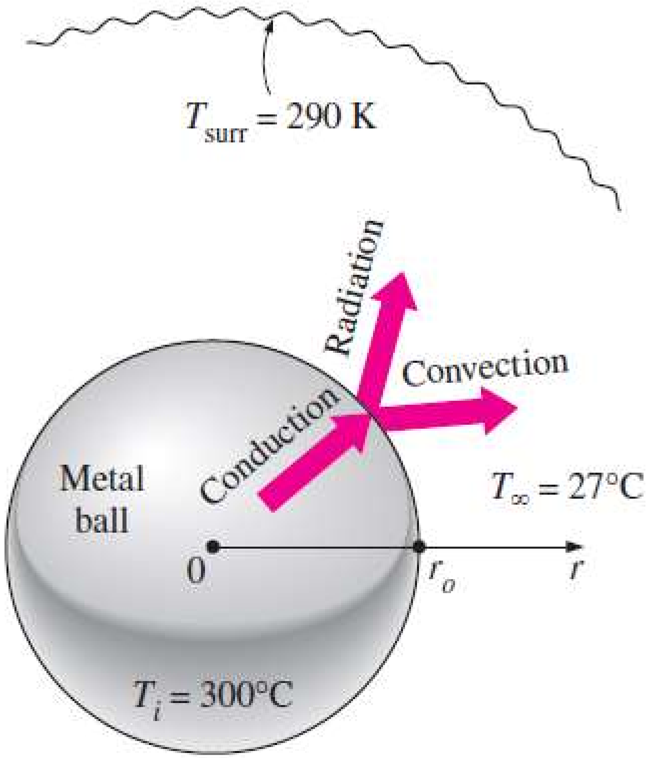
$$T_A(x_0, t) = T_B(x_0, t)$$

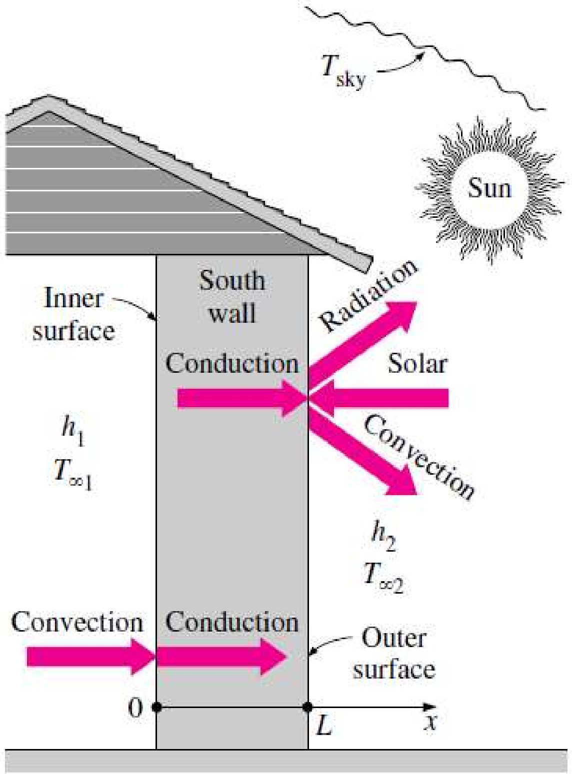
$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$



Generalized boundary condition

$$\left(\begin{array}{c}\text{Heat transfer} \\ \text{to the surface} \\ \text{in all modes}\end{array}\right) = \left(\begin{array}{c}\text{Heat transfer} \\ \text{from the surface} \\ \text{In all modes}\end{array}\right)$$



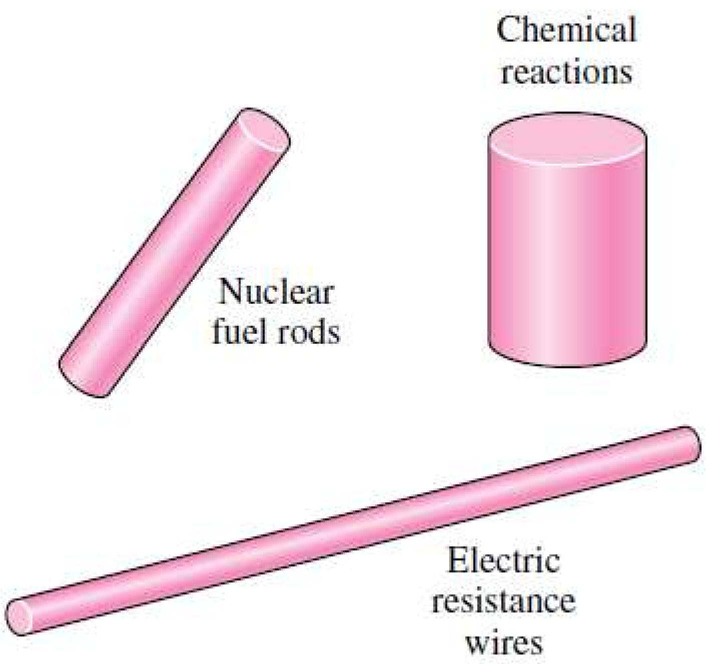


Example on Generalized boundary conditions:

Heat Generation in Solids

For example:

Resistance heating in wires,
exothermic chemical reactions in a solid,
nuclear reactions in nuclear fuel rods



$$\dot{e}_{gen} = \frac{\dot{E}_{gen,\ electric}}{V_{wire}} = \frac{I^2 R_e}{\pi r_o^2 L}$$

$$\begin{pmatrix} \text{Rate of} \\ \textit{heat transfer} \\ \text{from the solid} \end{pmatrix} = \begin{pmatrix} \text{Rate of} \\ \textit{energy generation} \\ \text{within the solid} \end{pmatrix}$$

$$\dot{Q} = \dot{e}_{gen} V \qquad \text{(W)}$$

Newton's law of cooling

$$\dot{Q} = hA_s(T_s - T_\infty) \qquad \text{(W)}$$

solving for the surface temperature T_s gives

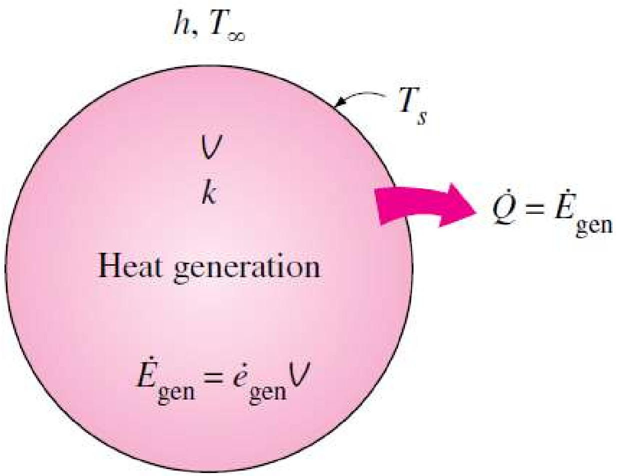
$$T_s = T_\infty + \frac{\dot{e}_{\text{gen}} V}{hA_s}$$

Long solid cylinder of radius r_o

$$A_s = 2\pi r_o L \text{ and } V = \pi r_o^2 L$$

solid sphere of radius r_o

$$A_s = 4\pi r_o^2 \text{ and } V = \tfrac{4}{3}\pi r_o^3$$



$$T_{s,\text{ plane wall}} = T_\infty + \frac{\dot{e}_{\text{gen}} L}{h}$$

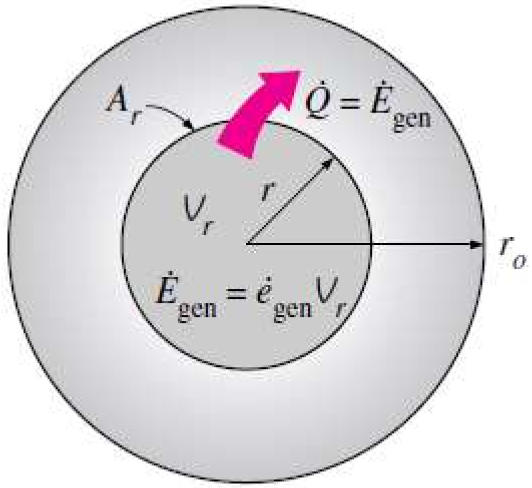
$$T_{s,\text{ cylinder}} = T_\infty + \frac{\dot{e}_{\text{gen}} r_o}{2h}$$

$$T_{s,\text{ sphere}} = T_\infty + \frac{\dot{e}_{\text{gen}} r_o}{3h}$$

Heat Generation in Solids -The maximum Temperature in a Cylinder (the Centerline)

The *heat generated* within an inner cylinder must be equal to the *heat conducted* through its outer surface.

$$-kA_r \frac{dT}{dr} = \dot{e}_{\text{gen}} V_r$$



Substituting these expressions into the above equation and separating the variables, we get

$$-k(2\pi r L)\frac{dT}{dr} = \dot{e}_{\text{gen}}(\pi r^2 L) \quad \rightarrow \quad dT = -\frac{\dot{e}_{\text{gen}}}{2k} r dr$$

Integrating from $r=0$ where $T(0)=T_0$ to $r=r_o$

Cylinder

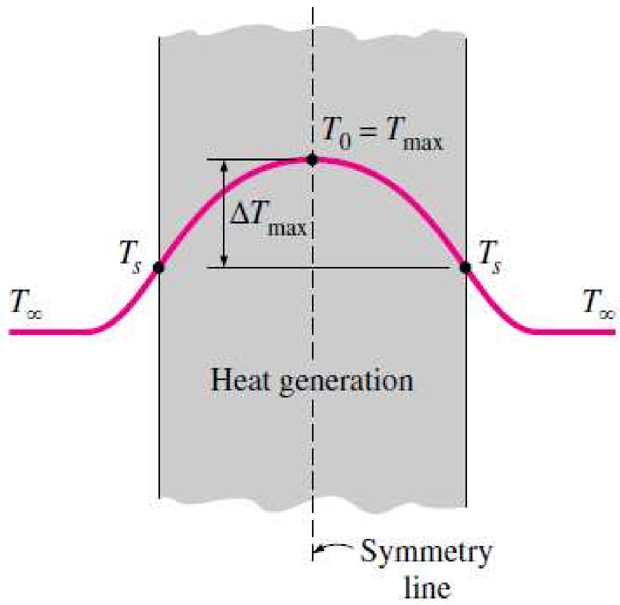
$$\Delta T_{\text{max, cylinder}} = T_0 - T_s = \frac{\dot{e}_{\text{gen}} r_o^2}{4k}$$

Plane wall

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{\text{gen}} L^2}{2k}$$

Sphere

$$\Delta T_{\text{max, sphere}} = \frac{\dot{e}_{\text{gen}} r_o^2}{6k}$$

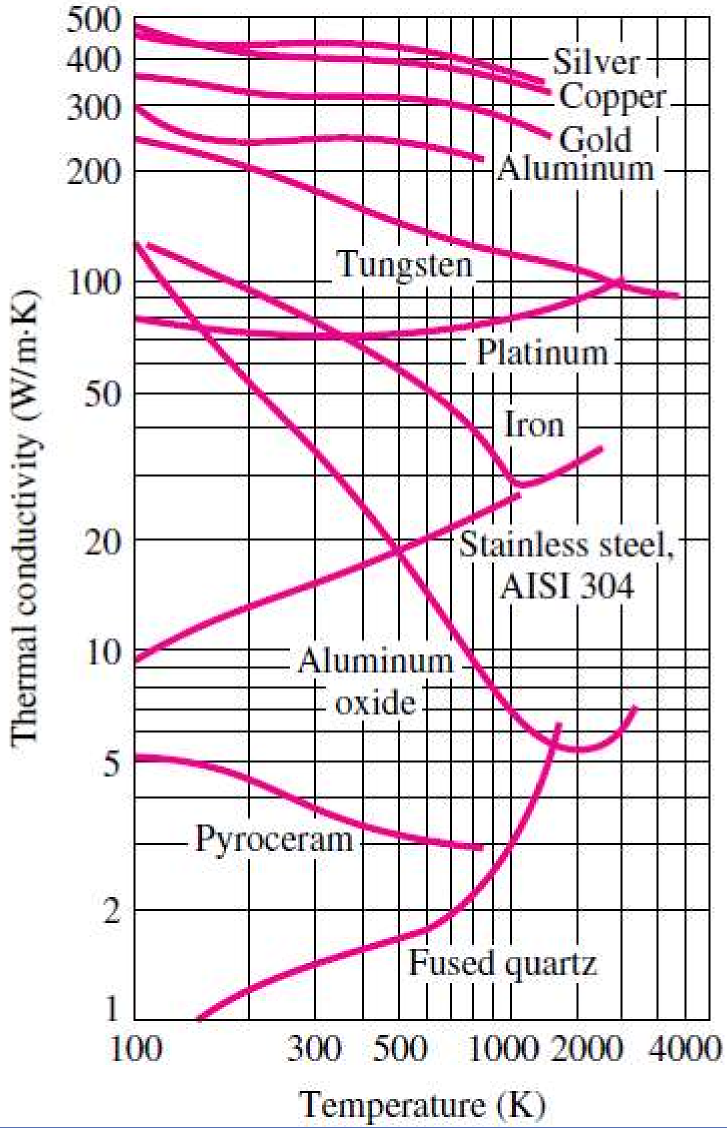


VARİABLE THERMAL CONDUCTİVİTY, *K*(*T*)

The thermal conductivity of a material, in general, varies with temperature.

An average value for the thermal conductivity is commonly used when the variation is mild.

This is also common practice for other temperature-dependent properties such as the density and specific heat.



Variation of the thermal conductivity of some solids with temperature.

VARİABLE THERMAL CONDUCTİVİTY FOR 1-D CASES

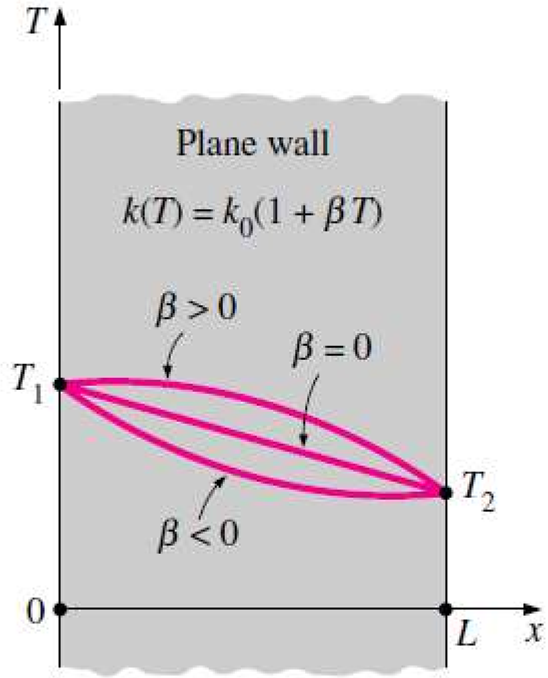
When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{avg} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

The variation in thermal conductivity of a material with can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

β is the **temperature coefficient of thermal conductivity**.



For a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant.
This is no longer the case when the thermal conductivity changes with temperature (even linearly).

Concluding Points

- One-Dimensional Heat Conduction
- General Heat Conduction Equation
- Boundary and Initial Conditions
- Solution of Steady One-Dimensional Heat Conduction Problems
- Heat Generation in a Solid
- Variable Thermal Conductivity $k(T)$