YILDIZ TECHNICAL UNIVERSITY
CIVIL ENGINEERING DEPARTMENT
DIVISION OF MECHANICS

STATICS

SECOND CHAPTER

VECTORS AND FORCES

Definitions : Quantities like mass, length, time, density of a body and number which have only magnitude are called as scalars. For investigating mechanical problems, definition of scalars only is not sufficient. So, in addition to scalars, there is need to define vectors.

Vector Quantities: Velocity, acceleration, and force which have sense, line of action and magnitude are called as vectors. The magnitude of a vector $\vec{F}$ is shown as $|\vec{F}|$ or F.

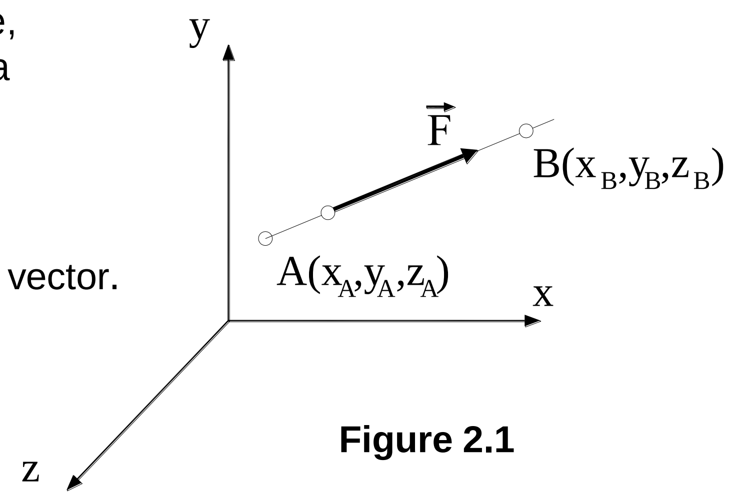


Figure 2.1

$A(X_A,Y_A,Z_A)$ and $B(X_B,Y_B,Z_B)$ are two points on the line of action of a vector.

Vectors can be classified into four types:

1)Free Vector : A free vector has specific magnitude, slope, and sense but its line of action does not pass through a unique point in space.

2)Sliding Vector: A sliding vector has a specific magnitude , slope, and sense and its line of action passes through a unique point in space. The point of application of a sliding vector can be anywhere along its line of action,Fig.2.2.

Lets consider the force $\vec{F}$ at the point A as seen in Figure 2.2a. By using the equilibrium principle, we can add two forces which are in equilibrium to the point B whose lines of action are same with the line of action of $\vec{F}$, Figure 2.2b.

By using superposition principle, the force $\vec{F}$ at A and the force $-\vec{F}$ at B can be removed. As a result, the force $\vec{F}$ affecting at the point A can be transferred to the point B, Figure 2.2c.

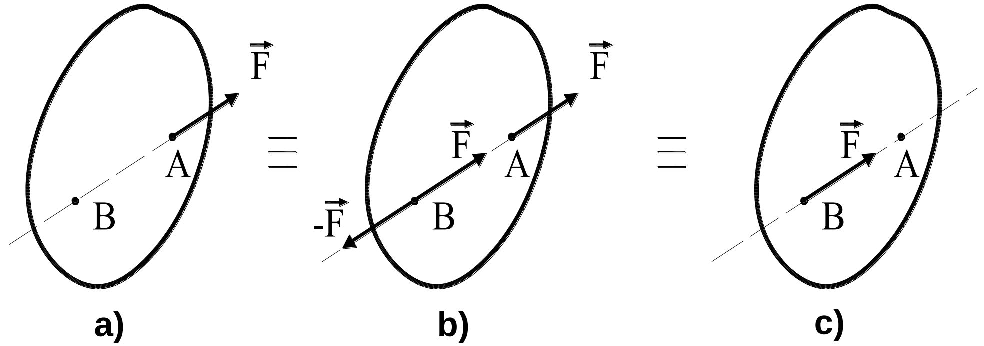


Fig. 2.2

3) Fixed Vector: A fixed vector has a specific magnitude ,slope ,and sense and its line of action passes through a unique point in space. The point of application of a fixed vector is confined to a fixed point on its line of action.

4)Unit Vector: Unit vectors are vectors having unit length and are denoted by $\vec{\lambda}$ and their magnitudes are $\left(\left|\vec{\lambda}\right| = 1\right)$, Fig.2.3. Rectangular unit vectors i , j, k are unit vectors having the direction of the positive x , y and z axes of a rectangular coordinate system. Here , the right–handed rectangular coordinate systems will be used. A unit vector can be specified by providing three angles as shown in Fig.2.3.

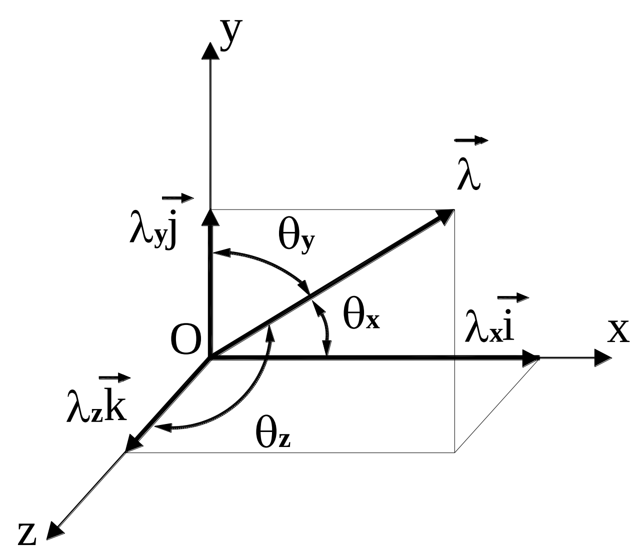


Figure 2.3

2.2.Vector operations and Forces: In statics, although whole operations can be made by using vectors, it is especially preferred in three dimensional problems for vector operations give the results more easily than the scalar operations. In this stage, some vector operations which will be used in statics, will be introduced :

Parallelogram Principle: Forces are added according to the parallelogram principle, Fig.2.4. In mathematical form this principle is shown as

$$\vec{R} = \vec{F}_1 + \vec{F}_2 \qquad \textbf{(2.3)}$$

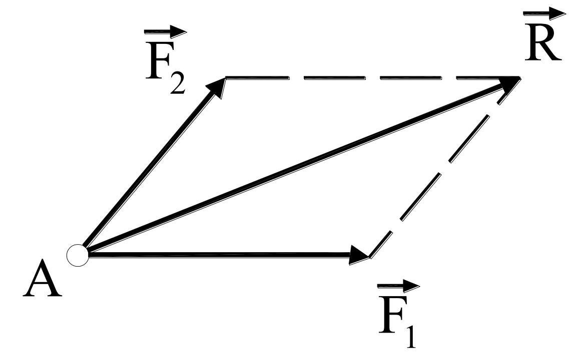


Fig. 2.4

Triangle Rule: The sum of the two vectors can be found by arranging $\vec{F}_1$ and $\vec{F}_2$ in tip-to-tail fashion and then connecting the tail of $\vec{F}_1$ with the tip of $\vec{F}_2$, Fig.2.5. The same result can be found by arranging $\vec{F}_2$ and $\vec{F}_1$ in tip-to-tail fashion and then connecting the tail of $\vec{F}_1$ with the tip of $\vec{F}_1$. This confirms the fact that vector addition is commutative.

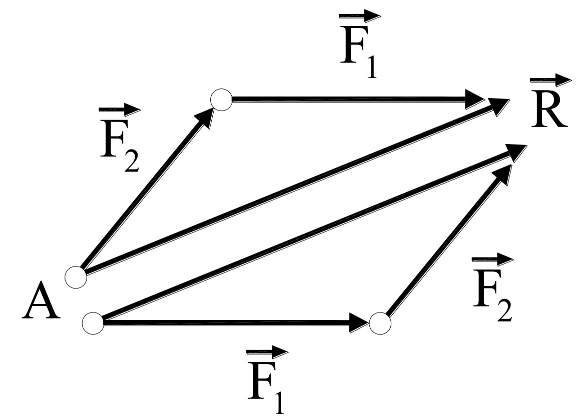


Fig. 2.5

Multiplication of Forces with a constant: If a vector F exerted on the point A is multiplied with a scalar,

$$a > 0 \rightarrow \vec{P} = a\vec{F} \qquad \textbf{(2.4)}$$

$$\vec{Q} = -a\vec{F} \qquad \textbf{(2.4)}$$

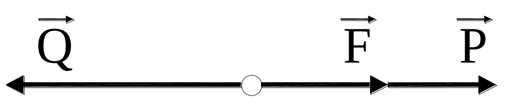


Fig. 2.6

- **Vector designation of forces:** It was defined that a force was a vector with its application point, magnitude, line of action and sense. Rectangular components of a force are F_x, F_y and F_z. In this situation, the force is

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \qquad \textbf{(2.5)}$$

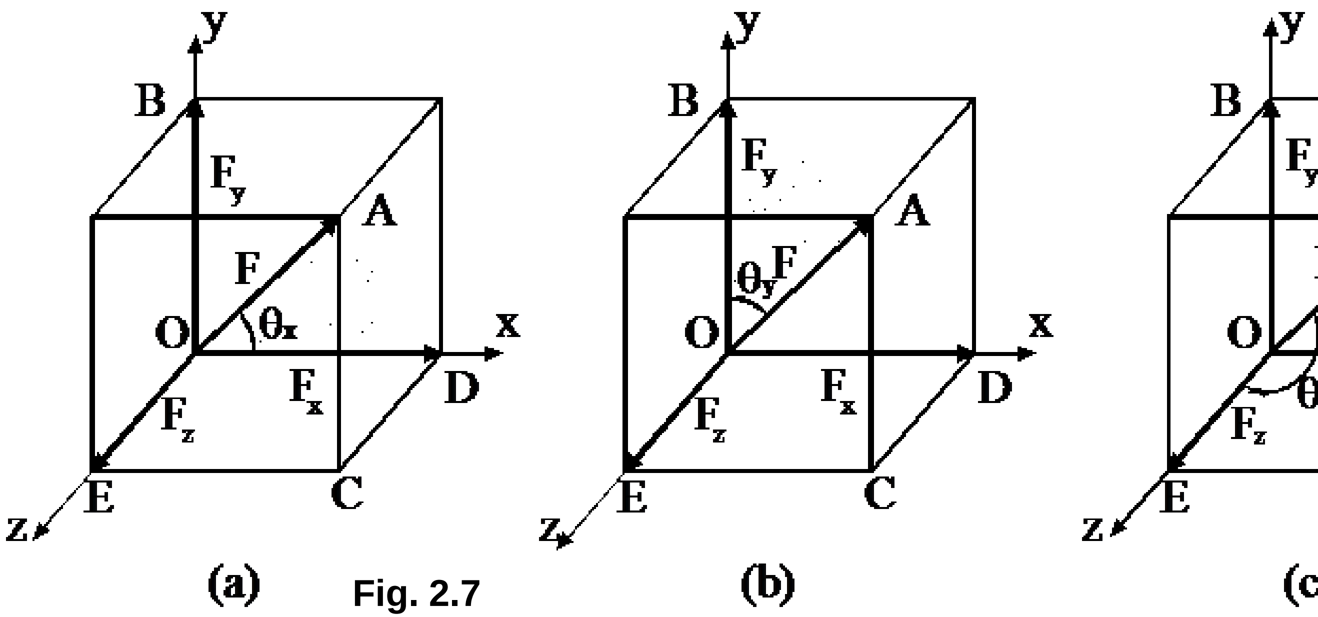


Fig. 2.7

From Fig. (2.7)

$$\left.\begin{aligned} F_x \vec{i} &= F.\cos\theta_x.\vec{i} \\ F_y \vec{j} &= F.\cos\theta_y.\vec{j} \\ F_z \vec{k} &= F.\cos\theta_z.\vec{k} \end{aligned}\right\} \qquad \textbf{(2.6)}$$

$$\vec{F} = F(\cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}) \qquad \textbf{(2.7)}$$

The cosines of $\theta_x, \theta_y, \theta_z$ are known as the direction cosines of the force $\vec{F}$, Fig.2.8. and these are shown as follows:

$$\lambda_x = \cos\theta_x \quad \lambda_y = \cos\theta_y \qquad \lambda_z = \cos\theta_z \tag{2.8}$$

$$\vec{\lambda} = \lambda_x \vec{i} + \lambda_y \vec{j} + \lambda_z \vec{k} \tag{2.9}$$

In this situation , the force $\vec{F}$ can be written as

$$\vec{F} = F\,\vec{\lambda} \tag{2.10}$$

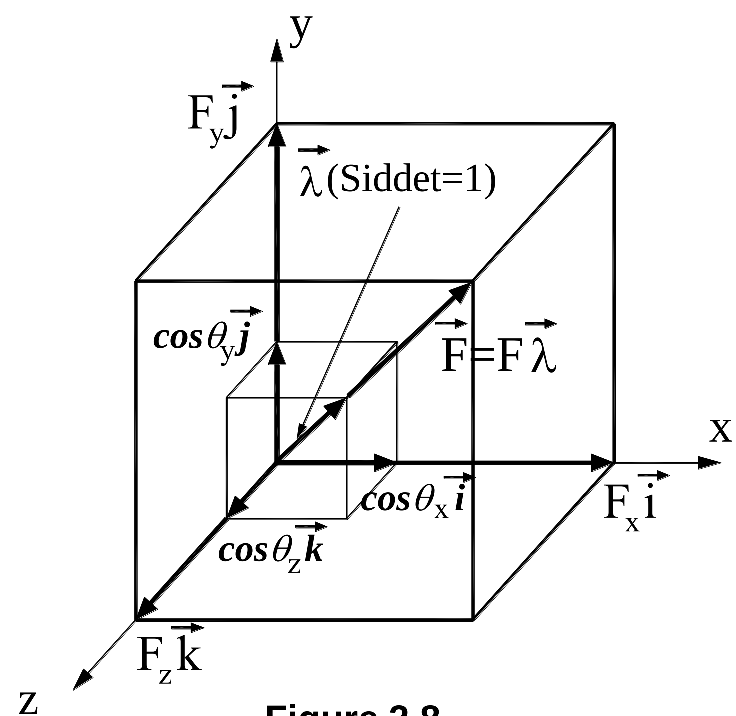


Figure 2.8

F is the magnitude of force $\vec{F}$ and the vector $\vec{\lambda}$ is referred to as the unit vector along the line of action of $\vec{F}$. Following equations are valid for unit vector $\vec{\lambda}$:

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

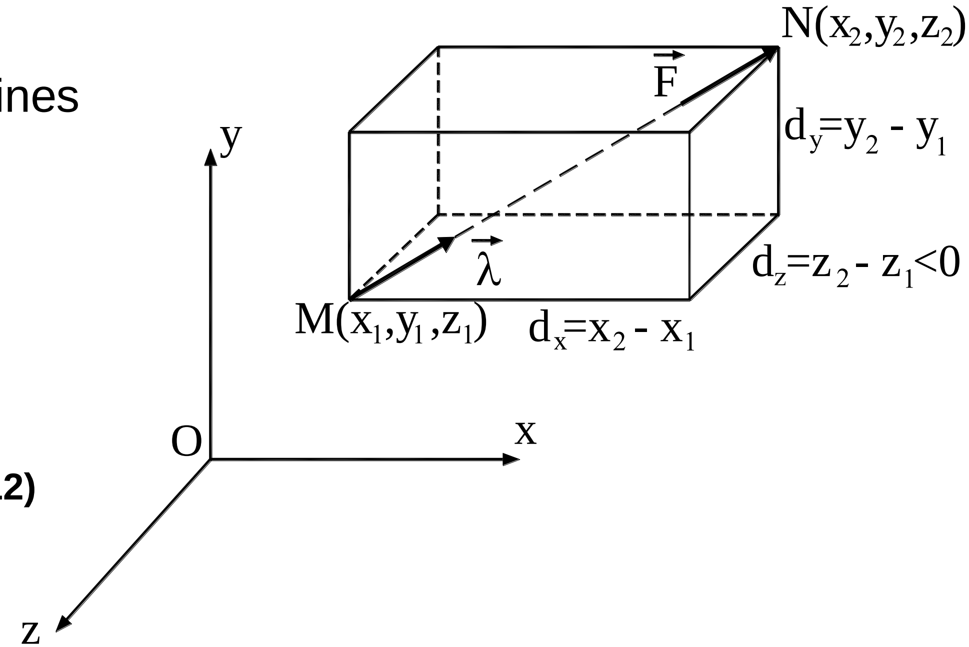
$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \tag{2.11}$$

$$\left|\vec{\lambda}\right| = 1 \qquad \left|\vec{\lambda}\right| = \vec{\lambda}\cdot\vec{\lambda} = \sqrt{\lambda_x^2 + \lambda_y^2 + \lambda_z^2}$$

Force defined by its magnitude and two points on its line of action: In many applications, the direction of a force (line of action and sense),is defined by the coordinates of two points, $M(x_1, y_1, z_1)$ and $N(x_2, y_2, z_2)$, located on the line of action of $\vec{F}$,Fig.2.9. Consider the vector MN joining M and N and of the same sense as $\vec{F}$. From Eq .(2.7), this force can be written by using its components as follows:

$$\vec{F} = F(\cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k})$$

From Fig.2.9., the direction cosines $\cos\theta_x, \cos\theta_y, \cos\theta_z$ can be obtained as



Şekil 2.9

$$\vec{\lambda}_{MN} = \frac{d x\vec{i} + dy\vec{j} + d z\vec{k}}{d}$$

$$\vec{\lambda}_{MN} = \frac{(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \quad \textbf{(2.12)}$$

$$\cos\theta_x = \frac{d_x}{d}\ ;\ \cos\theta_y = \frac{d_y}{d}\ ;\ \cos\theta_z = \frac{d_z}{d} \quad \textbf{(2.13)}$$

Now the force F can be written as follows:

$$\vec{F} = F.\vec{\lambda} = F\left(\lambda_x \vec{i} + \lambda_y \vec{j} + \lambda_z \vec{k}\right) = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \qquad \textbf{(2.14)}$$

The magnitude of the force F is $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$ **(2.15)**

Vector Product of two vectors: In mathematics, the vector product of $\vec{P}$ and $\vec{Q}$ is defined as the vector $\vec{V}$ which satisfies the following conditions:

$$\vec{V} = \vec{P} \times \vec{Q} \qquad \textbf{(2.16)}$$

a) The line of action $\vec{V}$ is perpendicular to the plane containing $\vec{P}$ and $\vec{Q}$, Fig.2.10. The sense of the resulting vector can be defined by using right hand rule: While four fingers of right hand is rotating from $\vec{P}$ to $\vec{Q}$, the thumb of right hand designates the sense of $\vec{V}$

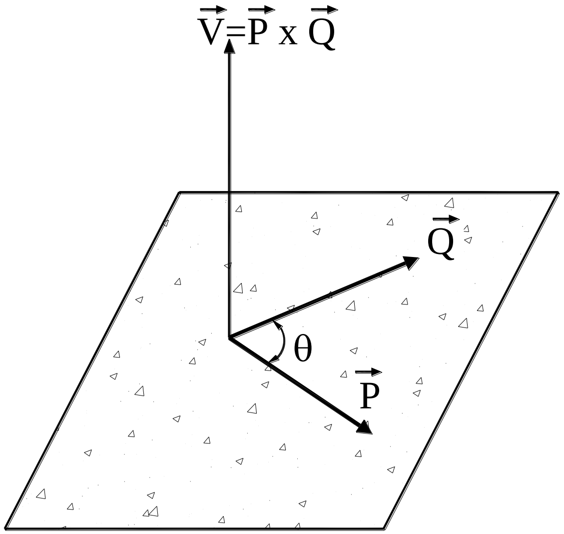


Figure 2.10

b) Magnitude of $\vec{V}$ is

$$V = P.Q.\sin\theta\left(\theta \leq 180^o\right) \qquad \textbf{(2.17)}$$

c)The distributive property holds:

$$\vec{P} \times (\vec{Q}_1 + \vec{Q}_2) = \vec{P} \times \vec{Q}_1 + \vec{P} \times \vec{Q}_2 \qquad \textbf{(2.18)}$$

d)The associative property does not apply to vector products

$$\vec{P} \times (\vec{Q} \times \vec{S}) \neq (\vec{P} \times \vec{Q}) \times \vec{S} \qquad \textbf{(2.19)}$$

e)Vector products are not commutative:

$$\vec{P}\times\vec{Q}\neq\vec{Q}\times\vec{P} \qquad \textbf{(2.20)}$$

f) The vector product $\vec{P}\times\vec{Q}$ remains unchanged if $\vec{Q}$ is replaced by a vector $\vec{Q}'$ which is coplanar with $\vec{P}$ and $\vec{Q}$ and such that the line joining the ties of $\vec{Q}$ and $\vec{Q}'$ is parallel to $\vec{P}$. Fig. **2.11.**

$$\vec{P}\times\vec{Q}=\vec{P}\times\vec{Q}' \qquad \textbf{(2.21)}$$

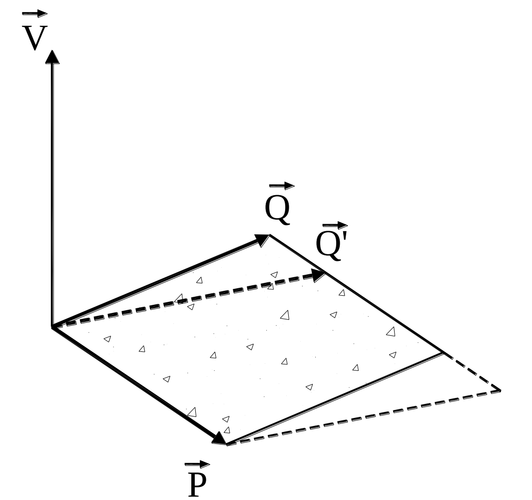


Fig. 2.11

Vector products expressed in terms of rectangular components:

Now,the vector product of any two of the unit vectors $\vec{i},\vec{j},\vec{k}$ which were defined in section 2.1, will be determined. Firstly, the product $\vec{i}\times\vec{j}$, Fig.2.12a, is considered. Since both vectors have magnitude equal to 1 and since they are at a right angle to each other, their vector product will also be a unit vector. This unit vector must be $\vec{k}$, since the vectors $\vec{i},\vec{j},\vec{k}$ are mutually perpendicular and form a right–handed triad. On the other hand, it follows from the right hand rule, the product $\vec{j}\times\vec{i}$ will be equal to $-\vec{k}$, Fig.2.12b. Finally, it should be observed that the vector product of a unit vector with itself , such as $\vec{i}\times\vec{i}$ is equal to zero, since both vectors have the same direction . The vector products of the various possible pairs of unit vectors are

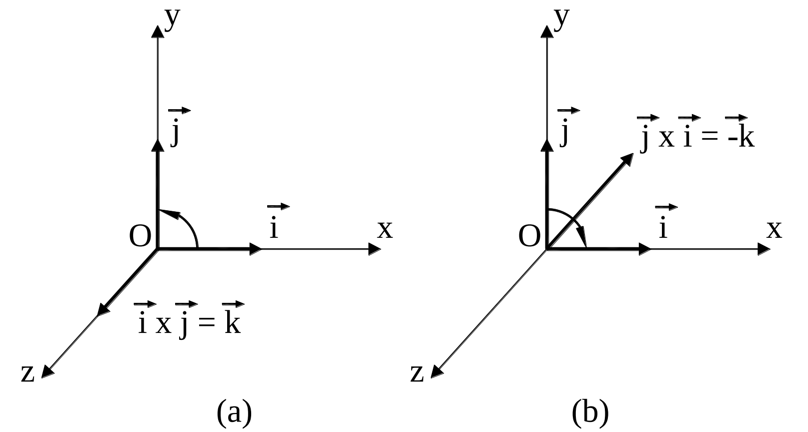


Fig. 2.12

$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i} \qquad \textbf{(2.22)}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$

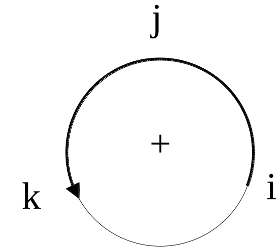


Fig. 2.13

We determine easily the sign of the vector product of two unit vectors in the following manner: We arrange the three letters i, j, k representing the unit vectors outside a circle and in the counterclockwise order. If two unit vectors follow each other in the counterclockwise order, then the vector product of these vectors will be positive, otherwise it will be negative. Fig.2.13.

The vector product $\vec{V}$ of two given vectors $\vec{P}$ and $\vec{Q}$ can easily be expressed in terms of the rectangular components of these vectors. Resolving $\vec{P}$ and $\vec{Q}$ into components, the following expression may be written:

$$\vec{V}=\vec{P}\times\vec{Q} \longrightarrow \vec{V}=\left(P_x\,\vec{i}+P_y\,\vec{j}+P_z\,\vec{k}\right)\times\left(Q_x\,\vec{i}+Q_y\,\vec{j}+Q_z\,\vec{k}\right)$$

Making use of the distributive property and recalling the identidies (2.23), after factoring out $\vec{i},\vec{j}$ and $\vec{k}$, the following expressions are obtained :

$$\vec{V}=P_xQ_y\,\vec{k}-P_xQ_z\,\vec{j}-P_yQ_x\,\vec{k}+P_yQ_z\,\vec{i}+P_zQ_x\,\vec{j}-P_zQ_y\,\vec{i} \qquad \textbf{(2.23)}$$

The rectangular components of the vector product $\vec{V}$ are thus found to be

$$\vec{V}=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$

$$\begin{aligned}V_x&=P_yQ_z-P_zQ_y\\V_y&=P_zQ_x-P_xQ_z\\V_z&=P_xQ_y-P_yQ_x\end{aligned} \qquad \textbf{(2.24)}$$

$$\vec{V} = P_x Q_y \vec{k} - P_x Q_z \vec{j} - P_y Q_x \vec{k} + P_y Q_z \vec{i} + P_z Q_x \vec{j} - P_z Q_y \vec{i} \qquad \textbf{(2.23)}$$

The rectangular components of the vector product $\vec{V}$ are thus found to be

$$\vec{V} = \left(P_y Q_z - P_z Q_y\right).\vec{i} + \left(P_z Q_x - P_x Q_z\right).\vec{j} + \left(P_x Q_y - P_y Q_x\right).\vec{k}$$

$$\begin{aligned} V_x &= P_y Q_z - P_z Q_y \\ V_y &= P_z Q_x - P_x Q_z \\ V_z &= P_x Q_y - P_y Q_x \end{aligned} \qquad \textbf{(2.24)}$$

Returning to Eq.(2.24), it is observed that its right-hand member represents the expansion of a determinant.The vector product $\vec{V}$ can thus be expressed in the following form, which is more easily memorized:

$$\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \qquad \textbf{(2.25)}$$

Scalar product of two vectors: The scalar product of two vectors $\vec{P}$ and $\vec{Q}$ is defined as the product of the magnitudes of $\vec{P}$ and $\vec{Q}$ and of the cosine of the angle α formed by $\vec{P}$ and $\vec{Q}$, Fig.2.14. This can be written as follows:

$$\vec{P}.\vec{Q} = P.Q.\cos\alpha \qquad \textbf{(2.26)}$$

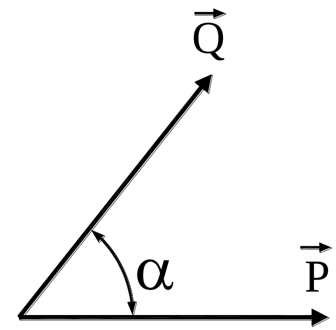


Fig. 2.14

Note that the expression just defined is not a vector but a scalar, which explains the name scalar product; because of the notation used, $\vec{P}\cdot\vec{Q}$ is also referred to as the dot product of the vectors $\vec{P}$ and $\vec{Q}$.

a) It follows from its very definition that the scalar product of two vectors is commutative:

$$\vec{P}.\vec{Q} = \vec{Q}.\vec{P} \qquad \textbf{(2.27)}$$

b) Scalar product is distributive (2.28):

$$\vec{P}.(\vec{Q}_1 + \vec{Q}_2) = \vec{P}.\vec{Q}_1 + \vec{P}.\vec{Q}_2 \qquad \textbf{(2.28)}$$

c) As far as the third property, the associative property is concerned, it should be noted that this property cannot apply to scalar products. Indeed, $\left(\vec{P}\cdot\vec{Q}\right)\cdot\vec{S}$ has no meaning, since $\left(\vec{P}\cdot\vec{Q}\right)$ is not a vector but a scalar.

The scalar product of two vectors $\vec{P}$ and $\vec{Q}$ can be expressed in terms of their rectangular components. Resolving $\vec{P}$ and $\vec{Q}$ into components, the following expression is written first:

$$\vec{P} \cdot \vec{Q} = \left(P_x \vec{i} + P_y \vec{j} + P_z \vec{k}\right) . \left(Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k}\right) \qquad \textbf{(2.29)}$$

Making use of the distributive property, $\vec{P} \, . \, \vec{Q}$ is expressed as the sum of scalar products, such as $P_x \vec{i} \cdot Q_x \vec{i}$ and $P_x \vec{i} \cdot Q_y \vec{j}$. However, from the definition of the scalar product, it follows that the scalar products of the unit vectors are either zero or one:

$$\begin{array}{lll} \vec{i} \cdot \vec{i} = 1 & \vec{j} \cdot \vec{j} = 1 & \vec{k} \cdot \vec{k} = 1 \\ \vec{i} \cdot \vec{j} = 0 & \vec{j} \cdot \vec{k} = 0 & \vec{k} \cdot \vec{i} = 0 \end{array} \qquad \textbf{(2.30)}$$

Thus, the expression obtained for $\vec{P} \, . \, \vec{Q}$ reduces to

$$\vec{P} \cdot \vec{Q} = P_x Q_x + P_v Q_v + P_z Q_z \qquad \textbf{(2.31)}$$

In particular case, when $\vec{P}$ and $\vec{Q}$ are equal, it should be noted that

$$\vec{P} \cdot \vec{P} = P_x^2 + P_y^2 + P_z^2 \qquad \textbf{(2.32)}$$

Angle formed by two given vectors: Let two vectors be given in terms of their components:

$$\vec{P} = P_x \vec{i} + P_y \vec{j} + P_z \vec{k} \qquad \vec{Q} = Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k} \qquad \textbf{(2.33)}$$

To determine the angle formed by two vectors, the expressions obtained in (2.26) and (2.31) are equated for their scalar product and the following is written:

$$P \, Q \cos\theta = P_x Q_x + P_y Q_y + P_z Q_z \qquad \textbf{(2.34)}$$

Solving for $\cos\theta$, the following expression is obtained: $\cos\theta = \dfrac{P_x . Q_x + P_y . Q_y + P_z . Q_z}{P . Q}$ **(2.35)**

Projection of a vector on a given axis: Consider a vector $\vec{P}$ forming an angle θ with an axis, or directed line, *OL*, Fig.2.15. The projection of $\vec{P}$ on the axis *OL* is defined as the scalar

$$P_{OL} = P.\cos\theta \qquad \textbf{(2.36)}$$

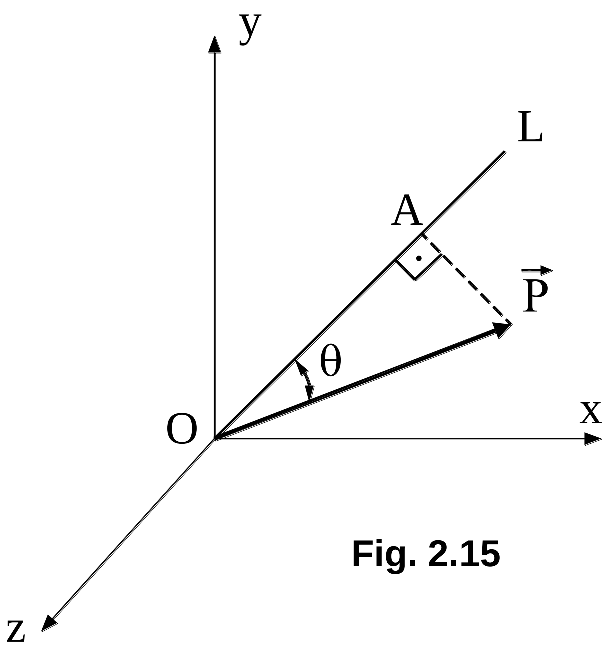


Fig. 2.15

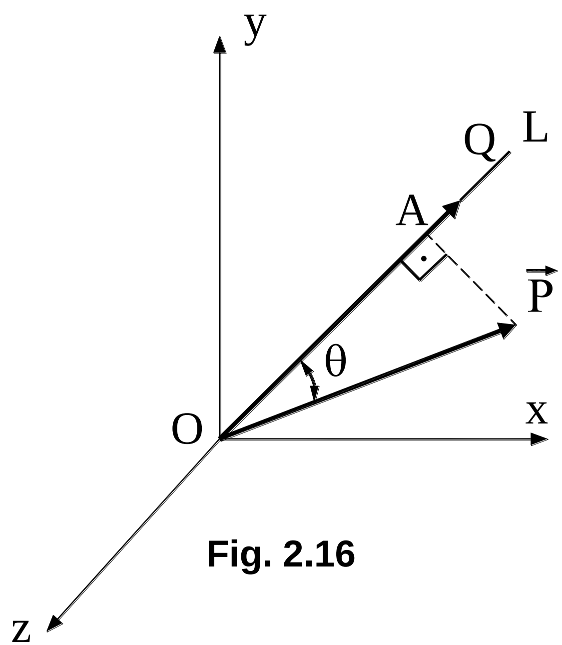


Fig. 2.16

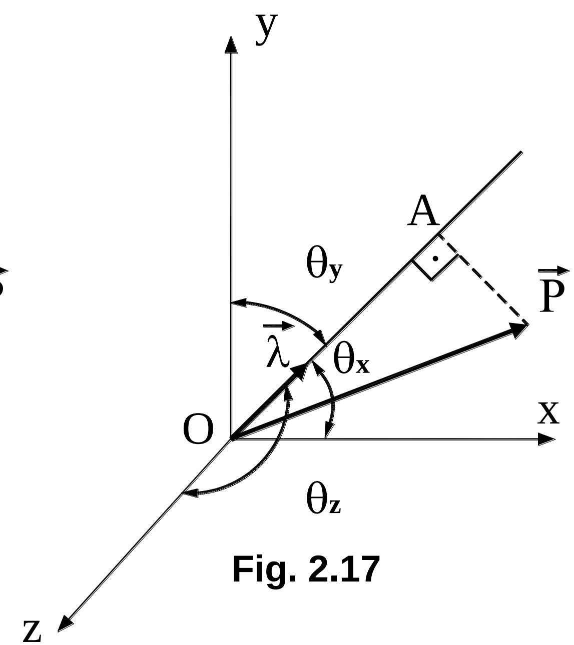


Fig. 2.17

The projection P_{OL} is equal in absolute value to the length of segment; it will be positive if *OA* has the same sense as the axis *OL*, that is if θ is acute, and negative otherwise. If $\vec{P}$ and *OL* are at a right angle, the projection of $\vec{P}$ on *OL* is zero.

Consider now a vector $\vec{Q}$ directed along *OL* and of the same sense as *OL*, Fig.2.16. The scalar product of $\vec{P}$ and $\vec{Q}$ can be expressed as

$$\vec{P}\cdot\vec{Q} = P\cdot Q\cos\theta = P_{OL}\,Q \qquad \textbf{(2.37)}$$

from which it follows that

$$P_{OL} = \frac{\vec{P}\cdot\vec{Q}}{Q} = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{Q} \qquad \textbf{(2.38)}$$

In the particular case when the vector selected along *OL* is the unit vector $\vec{\lambda}$, it is written as

$$P_{OL} = \vec{P}\cdot\vec{\lambda} \qquad \textbf{(2.39)}$$

Resolving $\vec{P}$ and $\vec{\lambda}$ into rectangular components of $\vec{\lambda}$ along the coordinate axis are respectively equal to the direction cosines of OL , the projection of P on OL is expressed as

$$P_{OL} = P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z \qquad \textbf{(2.40)}$$

where θ_x, θ_y and θ_z, denote the angles that the axis *OL* forms with the coordinate axes, Fig. 2.17.

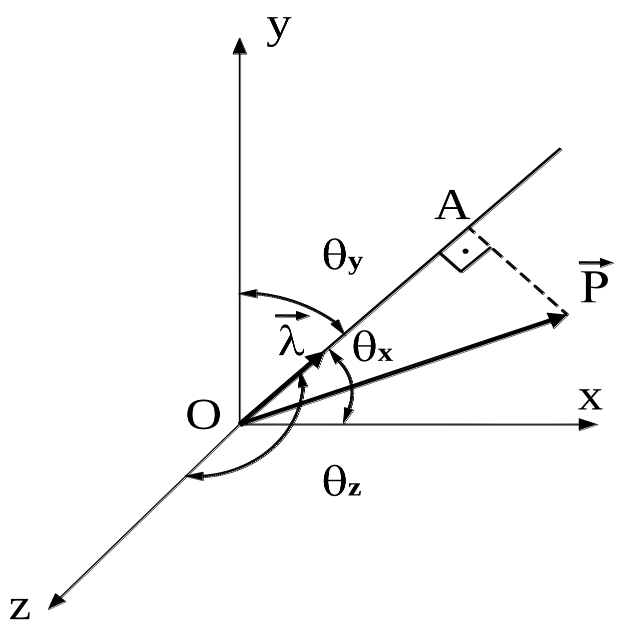
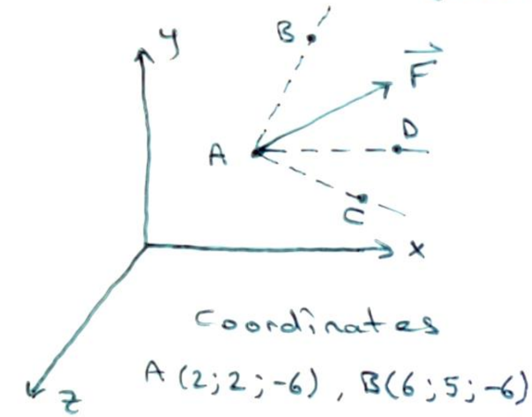


Fig. 2.17

Problem: Example Problem: Resolve the force $\vec{F}$ into its components in the AB, AC and AD directions.



$$\vec{\lambda}_{AB} = \frac{4\vec{i}+3\vec{j}}{5} \quad , \quad \vec{\lambda}_{AC} = \frac{4\vec{i}-3\vec{j}}{5}$$

$$\vec{\lambda}_{AD} = \frac{7\vec{i}+12\vec{k}}{13,89}$$

Coordinates

A (2;2;-6), B(6;5;-6)

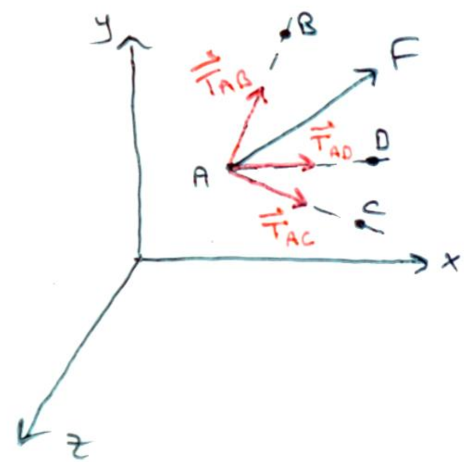
C(6;-1;-6), D(9;2;6)

$$\vec{F} = T_{AB}\cdot\vec{\lambda}_{AB} + T_{AC}\cdot\vec{\lambda}_{AC} + T_{AD}\cdot\vec{\lambda}_{AD}$$

$\vec{F} = 50\vec{i}+20\vec{j}+100\vec{k}$ N

$$50\vec{i}+20\vec{j}+100\vec{k} = T_{AB}\left(\frac{4\vec{i}+3\vec{j}}{5}\right)+T_{AC}\left(\frac{4\vec{i}-3\vec{j}}{5}\right)+$$

$$+\,T_{AD}\left(\frac{7\vec{i}+12\vec{k}}{13,89}\right)$$



$$50 = \frac{4}{5}T_{AB} + \frac{4}{5}T_{AC} + \frac{7}{13,89}T_{AD} \qquad (1)$$

$$20 = \frac{3}{5}T_{AB} - \frac{3}{5}T_{AC} \qquad (2)$$

$$100 = \frac{12}{13,89}T_{AD} \qquad (3)$$

From Eqs. (1), (2), (3)

$T_{AB} = 11,46$ N

$T_{AC} = -21,875$ N

$T_{AD} = 115,75$ N

Example Problem: A rectangular prism is acted upon by a force along the diagonal AB:

a) Determine the components of the force $\vec{F}$ along the diagonal AB