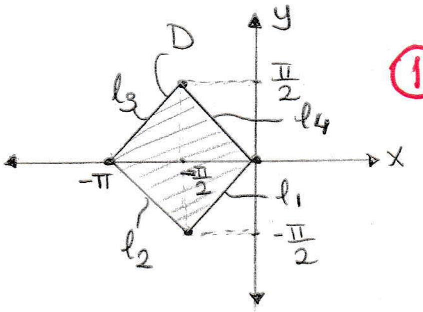
Soru 1: D bölgesi köşeleri $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $(-\pi, 0)$, $(0,0)$, $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ noktalarında bulunan kare olmak üzere,

$\iint_D (x+y)^2 \cdot \sin(y-x)\, dy\,dx$ integralini hesaplayınız. **(20P)**



Çözüm:

$\ell_1 : (0,0) \to \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$

$$\frac{y-0}{0-\left(-\frac{\pi}{2}\right)} = \frac{x-0}{0-\left(-\frac{\pi}{2}\right)} \Longrightarrow y = x$$

$\ell_2 : (-\pi, 0) \to \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$

$$\frac{y-0}{0-\left(-\frac{\pi}{2}\right)} = \frac{x-(-\pi)}{-\pi-\left(-\frac{\pi}{2}\right)} \Longrightarrow y = -x-\pi$$

$\ell_3 : (-\pi, 0) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

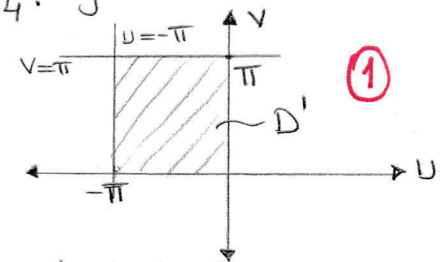
$$\frac{y-0}{0-\frac{\pi}{2}} = \frac{x-(-\pi)}{-\pi-\left(-\frac{\pi}{2}\right)} \Longrightarrow y = x+\pi$$

$\ell_4 : (0,0) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{y-0}{0-\frac{\pi}{2}} = \frac{x-0}{0-\left(-\frac{\pi}{2}\right)} \Longrightarrow y = -x$$

$$\left.\begin{aligned} x+y &= u \\ y-x &= v \end{aligned}\right]$$

D	D'
$\ell_1 : y-x=0 \longrightarrow$	$v=0$
$\ell_2 : y+x=-\pi \longrightarrow$	$u=-\pi$
$\ell_3 : y-x=\pi \longrightarrow$	$v=\pi$
$\ell_4 : y+x=0 \longrightarrow$	$u=0$



$$J(u,v) = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{1}{2} \qquad dx\,dy = \frac{1}{2}\, du\,dv$$

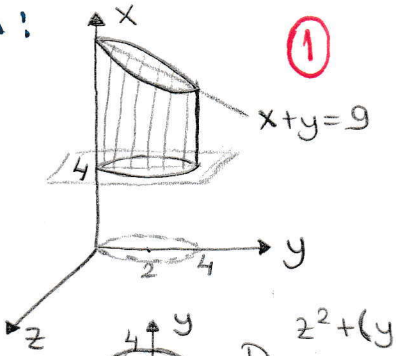
$$I = \iint_{D'} u^2 \cdot \sin v \cdot \frac{1}{2}\, du\,dv = \int_{u=-\pi}^{u=0} \int_{v=0}^{v=\pi} \frac{1}{2} \sin v \cdot u^2\, dv\,du$$

$$= \int_{-\pi}^{0} (-\cos v)\Big|_0^{\pi} \frac{1}{2} u^2\, du = \int_{-\pi}^{0} (1+1) \cdot \frac{1}{2}\, u^2\, du$$

$$= \frac{u^3}{3}\Big|_{-\pi}^{0} = \frac{\pi^3}{3} \quad //$$

Soru 2 : $z^2+(y-2)^2=4$, $x=4$ ve $x+y=9$ yüzeyleri ile oluşturulmuş kapalı cismin hacmini iki katlı kutupsal integral ile hesaplayın. (20P)

Çözüm :



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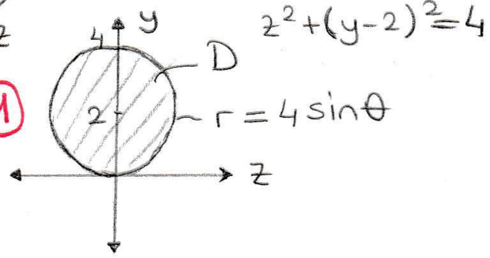

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$$V=\iint_D (x_1-x_2)\,dy\,dz$$

$$V=\iint_D (9-y-4)\,dy\,dz$$

$$V=\iint_D (5-y)\,dy\,dz \quad 2$$

$$z=r\cos\theta$$
$$y=r\sin\theta \quad 2$$
$$dy\,dz=r\,dr\,d\theta$$

$$V=\int_{\theta=0}^{\pi}\int_{r=0}^{4\sin\theta}(5-r\sin\theta)\,r\,dr\,d\theta=\int_0^{\pi}\int_0^{4\sin\theta}(5r-r^2\sin\theta)\,dr\,d\theta$$

(1) (1) (2) (1) (1)

$$=\int_0^{\pi}\left(\frac{5}{2}r^2-\frac{r^3}{3}\sin\theta\right)\Bigg|_0^{4\sin\theta}d\theta=\int_0^{\pi}\left(40\sin^2\theta-\frac{64}{3}\sin^4\theta\right)d\theta \quad 2$$

$$=\int_0^{\pi}\left[40\left(\frac{1-\cos2\theta}{2}\right)-\frac{64}{3}\left(\frac{1-\cos2\theta}{2}\right)^2\right]d\theta \quad 2$$

$$=\int_0^{\pi}\left[20-20\cos2\theta-\frac{16}{3}+\frac{32}{3}\cos2\theta-\frac{16}{3}\left(\frac{1+\cos4\theta}{2}\right)\right]d\theta$$

$$=\left[12\theta-\frac{14}{3}\sin2\theta-\frac{2}{3}\sin4\theta\right]\Bigg|_0^{\pi} \quad 2$$

$$=12\pi \text{ br}^3 \quad 2$$

Soru 3: $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{4}{2}, \frac{7}{4}, \frac{10}{8}, \frac{13}{16}, \frac{16}{32}, \dots \right\}$ dizisi verilsin.

Buna göre $\sum_{n=1}^{\infty} a_n$ serisinin toplamını hesaplayınız. (15P)

Çözüm: $\{a_n\} = \left\{ \frac{3n+1}{2^n} \right\}$ 1

$$\sum_{n=1}^{\infty} \frac{3n+1}{2^n} = \underbrace{\sum_{n=1}^{\infty} \frac{3n}{2^n}}_{A} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n}}_{B}$$ 2

$$B = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \Longrightarrow \begin{cases} a = \frac{1}{2} \\ r = \frac{1}{2} < 1 \end{cases}$$ Geometrik seri $\frac{a}{1-r}$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1,$$ 2 1

$$A = 3 \cdot \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (|x|<1)$$ 2

Türev alalım

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2} \quad (|x|<1)$$ 2

x ile çarpalım

$$\sum_{n=1}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2} \quad (|x|<1)$$ 2

$$A = 3 \cdot \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n = 3 \cdot \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} \quad \left(\frac{1}{2}<1\right)$$ 1

$$= 6,$$ 1

$$\sum_{n=1}^{\infty} \frac{3n+1}{2^n} = A + B = 6 + 1 = 7$$ 1

Soru 4: $\sum_{n=1}^{\infty} \left(1-\sqrt[n]{\ln 2}\right)$ serisinin karakterini belirleyiniz. (10P)

Çözüm:

$\sum_{n=1}^{\infty} \frac{1}{n}$ harmonik serisini (ıraksak) seçelim. 1

Limit karşılaştırma testini uygularsak,

$$\lim_{n\to\infty} \frac{1-(\ln 2)^{1/n}}{\frac{1}{n}} = \frac{0}{0} \quad 1$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}}\cdot \underbrace{(\ln 2)^{1/n}}_{1} \cdot \ln(\ln 2)}{-\cancel{\frac{1}{n^2}}} = -\ln(\ln 2) \neq 0, \infty$$

3 1 1

olduğundan iki seri aynı karakterdedir. 1

$\sum_{n=1}^{\infty} \frac{1}{n}$ ıraksak olduğundan,

$\sum_{n=1}^{\infty} \left(1-\sqrt[n]{\ln 2}\right)$ serisi ıraksaktır. 2

Soru 5: $P(1,2,2)$ bir nokta, $f(x,y,z)$ ve $g(x,y,z)$ aşağıdaki koşulları sağlayan, türevlenebilen iki fonksiyon olsun.

I) $f(P)=1$ ve $g(P)=4$

II) $\left.\frac{\partial g}{\partial z}\right|_P=-2$

III) f nin P noktasındaki en hızlı artışının gerçekleştiği yön $\vec{u}=4\vec{i}-\vec{j}+8\vec{k}$ ve bu yöndeki türevinin değeri 3 tür.

IV) $f(x,y,z)+3g(x,y,z)=13$ yüzeyinin P noktasındaki teğet düzleminin denklemi $x+4y+5z=19$ dur.

Buna göre $\left.\frac{\partial g}{\partial x}\right|_P$ değerini hesaplayınız. **(20P)**

Çözüm: $\vec{u}=\langle 4,-1,8\rangle \quad \nabla f|_P \parallel \vec{u}$

$$\nabla f|_P=\langle 4k,-k,8k\rangle \quad (2)$$

$$|\nabla f|_P|=\sqrt{16k^2+k^2+64k^2}=3 \quad (1) \Rightarrow 81k^2=9 \Rightarrow k=\frac{3}{9}=\frac{1}{3} \quad (1)$$

$$\nabla f|_P=\left\langle \frac{4}{3},-\frac{1}{3},\frac{8}{3}\right\rangle=\langle f_x|_P, f_y|_P, f_z|_P\rangle \quad (3)$$

$$H: f(x,y,z)+3g(x,y,z)-13=0$$

$$\nabla H|_P=\langle (f_x+3g_x)|_P, (f_y+3g_y)|_P, (f_z+3g_z)|_P\rangle \; (3) =\langle c,4c,5c\rangle \quad (2)$$

$$f_z|_P+3g_z|_P=5c$$

$$\frac{8}{3}+3\cdot(-2)=5c \quad (2) \Rightarrow -\frac{10}{3}=5c \Rightarrow c=-\frac{2}{3} \quad (2)$$

$$f_x|_P+3g_x|_P=c$$

$$\frac{4}{3}+3g_x|_P=-\frac{2}{3} \quad (2)$$

$$3g_x|_P=-\frac{2}{3}-\frac{4}{3}=-2 \Rightarrow \left.\frac{\partial g}{\partial x}\right|_P=-\frac{2}{3} \quad (2)$$

//

Soru 6: $g(2,4)=-4$, $g(2.02, 3.96) = -4.04$ şartlarını sağlayan bir $g(x,y)$ fonksiyonunu göz önüne alalım. Bu fonksiyonunun $P(2,4)$ noktasındaki $\vec{u}=-3\vec{i}+6\vec{j}$ yönündeki türevinin değerini yaklaşık olarak hesaplayınız. **(15P)**

Çözüm:

$$g(x,y) \approx L(x,y) = g(2,4) + g_x(2,4)(x-2) + g_y(2,4)(y-4) \quad 2$$

$$g(2.02, 3.96) \approx -4 + g_x(2,4)(2.02-2) + g_y(2,4)(3.96-4)$$

$$-4.04 \approx -4 + g_x(2,4)(0.02) + g_y(2,4)(-0.04) \quad 2$$

$$-0.04 \approx 0.02\left(g_x|_P - 2g_y|_P\right)$$

$$\Rightarrow g_x|_P - 2g_y|_P \approx -2 \quad 2$$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{\langle -3, 6\rangle}{3\sqrt{5}} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \quad 2$$

$$D_{\vec{u}}\, g|_P = \frac{\vec{u}}{|\vec{u}|} \cdot \overset{1}{\nabla g|_P} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \cdot \left\langle g_x|_P, g_y|_P \right\rangle \quad 2$$

$$= -\frac{1}{\sqrt{5}}\left(g_x|_P - 2g_y|_P\right)^{2} \approx \frac{2}{\sqrt{5}} \quad 2 \quad //$$

(*)

Not: $\vec{u}$ vektörünü birim yapmadan işleme devam edene (*) kısmından puan verilmeyecektir. Öğrenciye sorudan toplam 6 puan verilecektir.