Heat and Mass Transfer, 3rd Edition
Yunus A. Cengel
McGraw-Hill, New York, 2007

Chapter 2
Heat Conduction Equation

Y. Peles (RPI)
M. Kanoglu (Univ. Gaziantep)

Outline

- Introduction
- One-dimensional heat conduction equation
- General heat conduction equation
- Boundary and initial conditions
- Solution of one-dimensional heat conduction problems
- Heat generation in a solid
- Variable thermal conductivity
- Conclusions

Objectives

- To understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional.
- To obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case.
- To identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions.
- To solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux.
- To analyze one-dimensional heat conduction in solids that involve heat generation.
- To evaluate heat conduction in solids with temperature-dependent thermal conductivity.

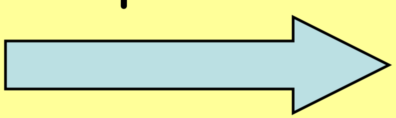
Introduction

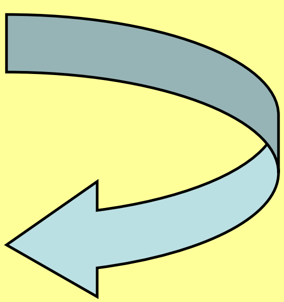
- Although heat transfer and temperature are closely related, they are of a different nature.
- Temperature has only magnitude

  it is a *scalar* quantity.
- Heat transfer has direction as well as magnitude

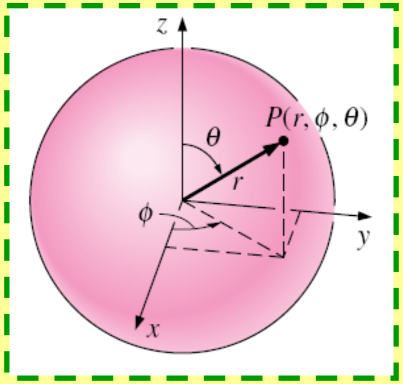
  

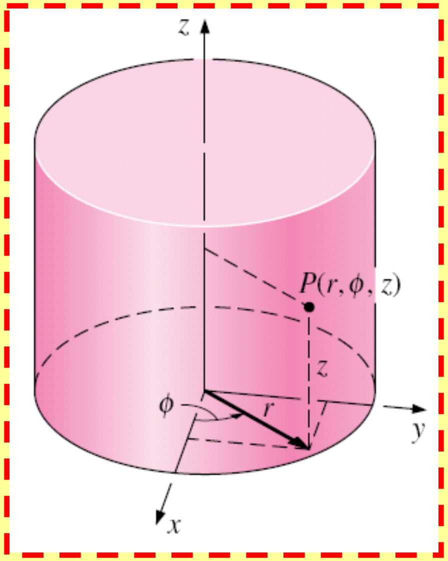
  it is a *vector* quantity.
- We work with a coordinate system and indicate direction with plus or minus signs.

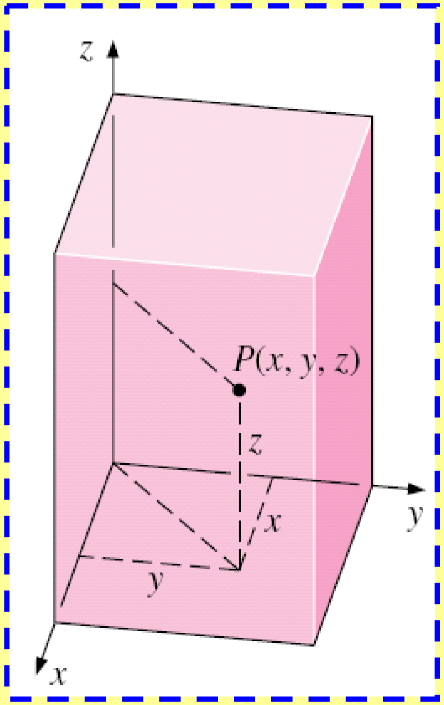




- The driving force for any form of heat transfer is the *temperature difference.*
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - rectangular (T(x, y, z, t)) ,
  - cylindrical (T(r, f, z, t)),
  - spherical (T(r, f, q, t)).



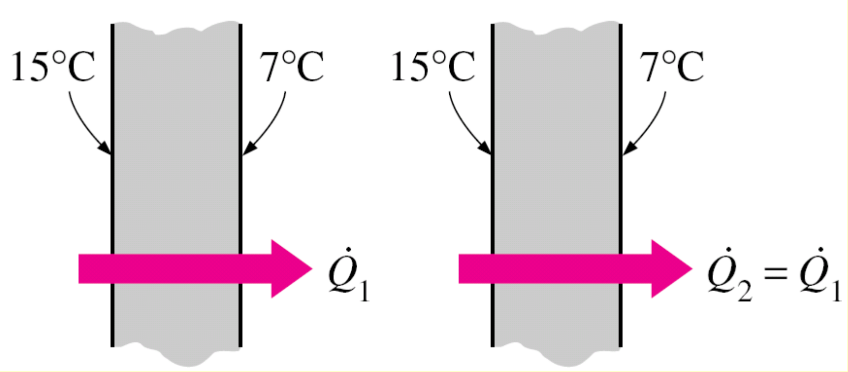




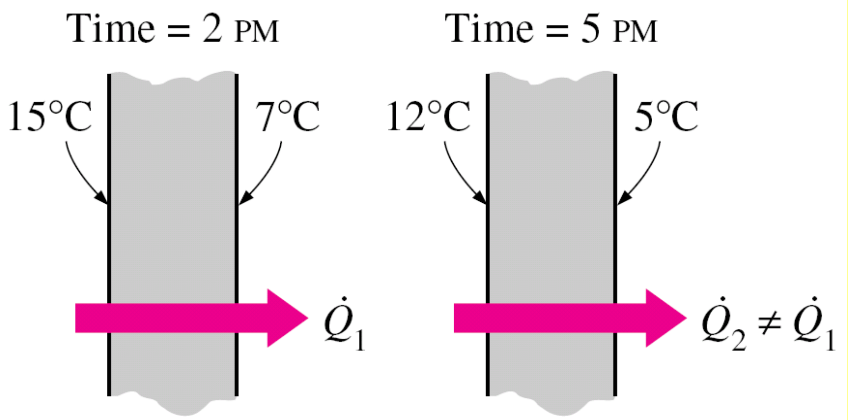
Classification of conduction heat transfer problems:

- steady versus transient heat transfer,
- multidimensional heat transfer,
- heat generation.

- ***Steady*** implies *no change* with time at any point within the medium



- ***Transient*** implies *variation with time* or *time dependence*

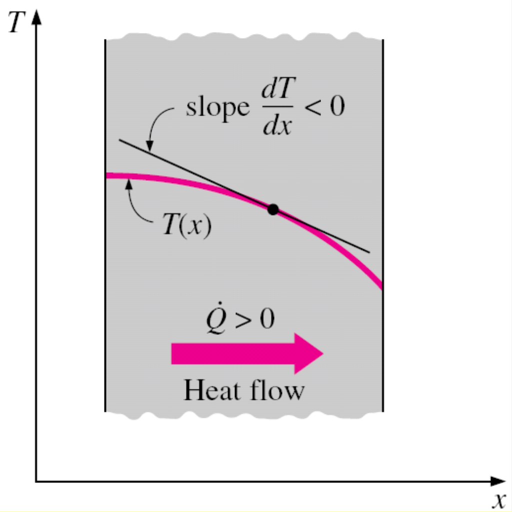


Multidimensional Heat Transfer

- Heat transfer problems are also classified as being:
  - *one-dimensional,*
  - *two dimensional,*
  - *three-dimensional.*
- In the most general case, heat transfer through a medium is **three-dimensional**. However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x* -direction.



Multidimensional Heat Transfer

- **One-dimensional** if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.
- **Two-dimensional** if the temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible.

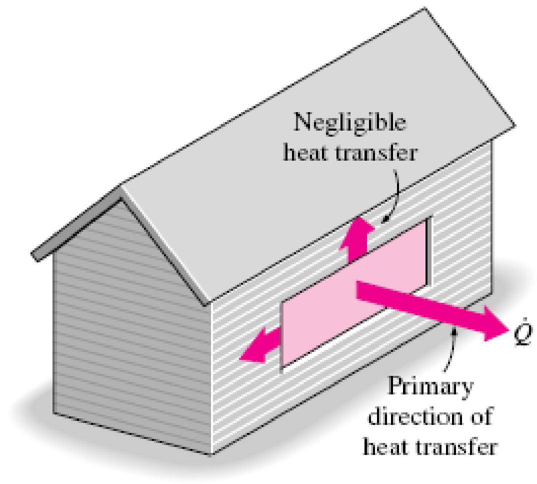


FIGURE 2–6
Heat transfer through the window of a house can be taken to be one-dimensional.

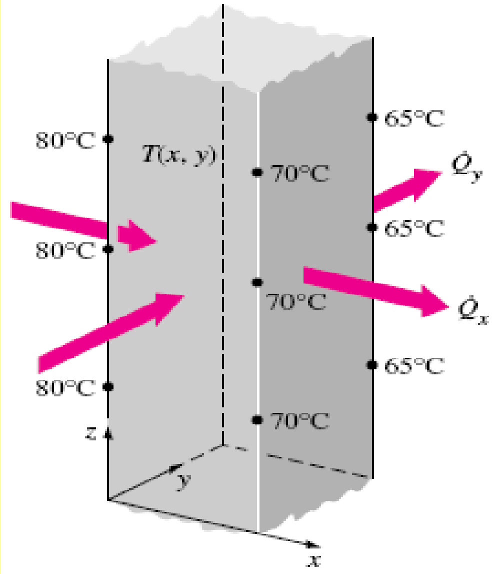


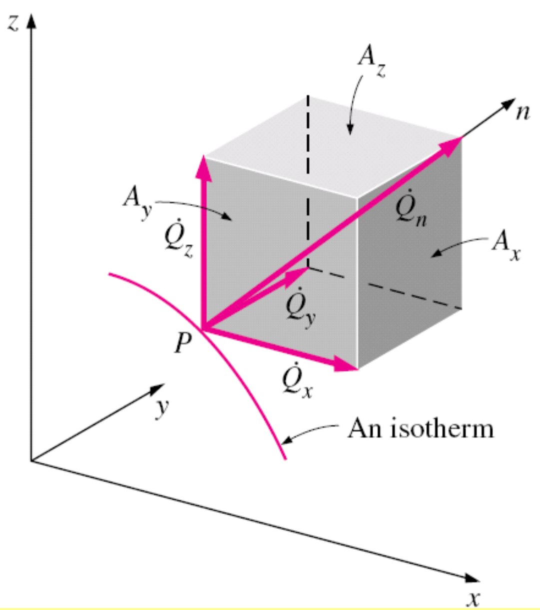
FIGURE 2–5
Two-dimensional heat transfer in a long rectangular bar.

General Relation for Fourier's Law of Heat Conduction

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal

surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{dT}{dn} \qquad \text{(W)}$$



In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

which can be determined from Fourier's law as

$$\begin{cases} \dot{Q}_x = -kA_x\dfrac{\partial T}{\partial x} \\ \dot{Q}_y = -kA_y\dfrac{\partial T}{\partial y} \\ \dot{Q}_z = -kA_z\dfrac{\partial T}{\partial z} \end{cases}$$

Heat Generation

- Examples:
  - electrical energy being converted to heat at a rate of I^2R,
  - fuel elements of nuclear reactors,
  - exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon.*
- The rate of heat generation units : W/m^3 or Btu/h · ft^3.
- The rate of heat generation in a medium may vary with time as well as position within the medium.
- The *total* rate of heat generation in a medium of volume V can be determined from

$$\dot{E}_{gen} = \int_V \dot{e}_{gen} dV \qquad \text{(W)}$$

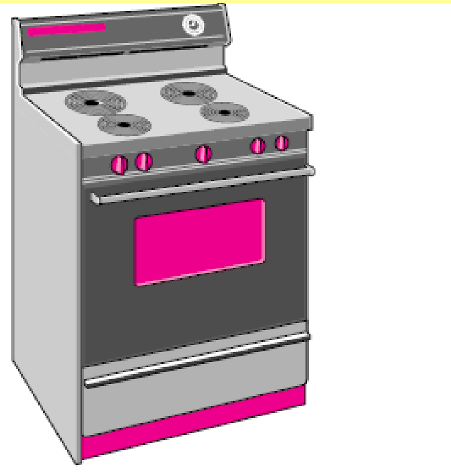


FIGURE 2–9
Heat is generated in the heating coils of an electric range as a result of the conversion of electrical energy to heat.

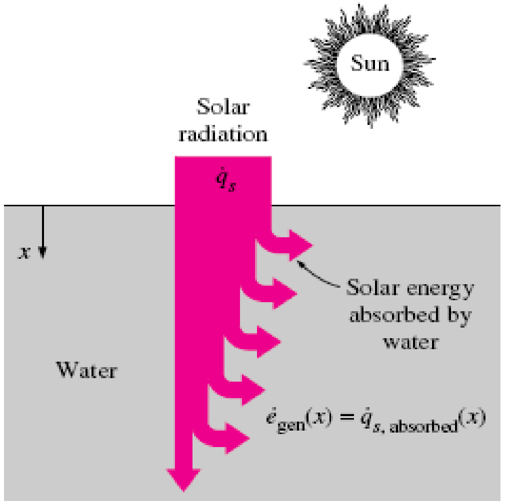


FIGURE 2–10
The absorption of solar radiation by water can be treated as heat generation.

EXAMPLE 2–2 **Heat Generation in a Hair Dryer**

The resistance wire of a 1200-W hair dryer is 80 cm long and has a diameter of $D = 0.3$ cm (Fig. 2–12). Determine the rate of heat generation in the wire per unit volume, in W/cm³, and the heat flux on the outer surface of the wire as a result of this heat generation.

SOLUTION The power consumed by the resistance wire of a hair dryer is given. The heat generation and the heat flux are to be determined.

Assumptions Heat is generated uniformly in the resistance wire.

Analysis A 1200-W hair dryer converts electrical energy into heat in the wire at a rate of 1200 W. Therefore, the rate of heat generation in a resistance wire is equal to the power consumption of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire,

$$\dot{e}_{\text{gen}} = \frac{\dot{E}_{\text{gen}}}{V_{\text{wire}}} = \frac{\dot{E}_{\text{gen}}}{(\pi D^2/4)L} = \frac{1200 \text{ W}}{[\pi(0.3 \text{ cm})^2/4](80 \text{ cm})} = 212 \text{ W/cm}^3$$

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire,

$$\dot{Q}_s = \frac{\dot{E}_{\text{gen}}}{A_{\text{wire}}} = \frac{\dot{E}_{\text{gen}}}{\pi DL} = \frac{1200 \text{ W}}{\pi(0.3 \text{ cm})(80 \text{ cm})} = 15.9 \text{ W/cm}^2$$

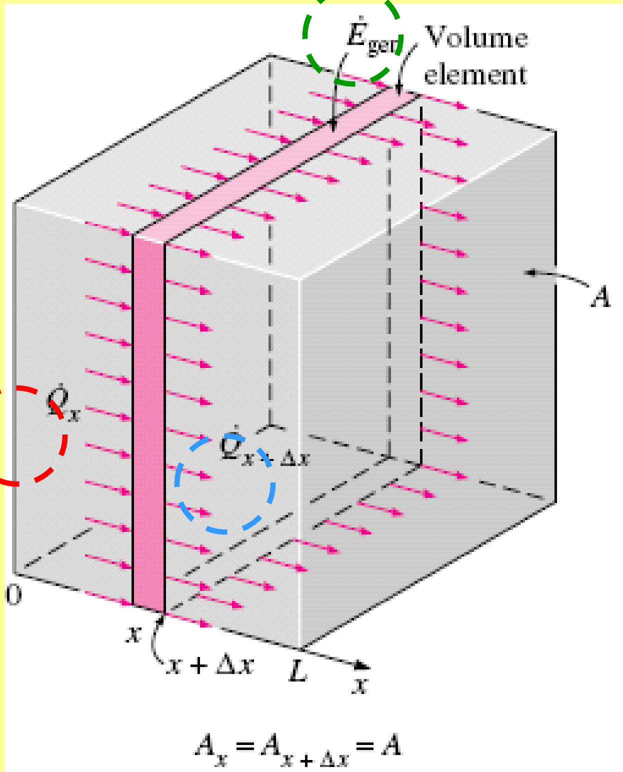
Discussion Note that heat generation is expressed per unit volume in W/cm³ or Btu/h · ft³, whereas heat flux is expressed per unit surface area in W/cm² or Btu/h · ft².

FIGURE 2–12
Schematic for Example 2–2.

One-Dimensional Heat Conduction Equation - Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta x\left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta x \end{cases}$$

- Substituting into above equation, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A\Delta x$, taking the limit as Δx→ 0 and Δt→ 0, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

The area A is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

Variable conductivity:
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity:
$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad ; \quad \alpha=\frac{k}{\rho c}$$

The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state:
$$\frac{d^2 T}{dx^2}+\frac{\dot{e}_{gen}}{k}=0$$

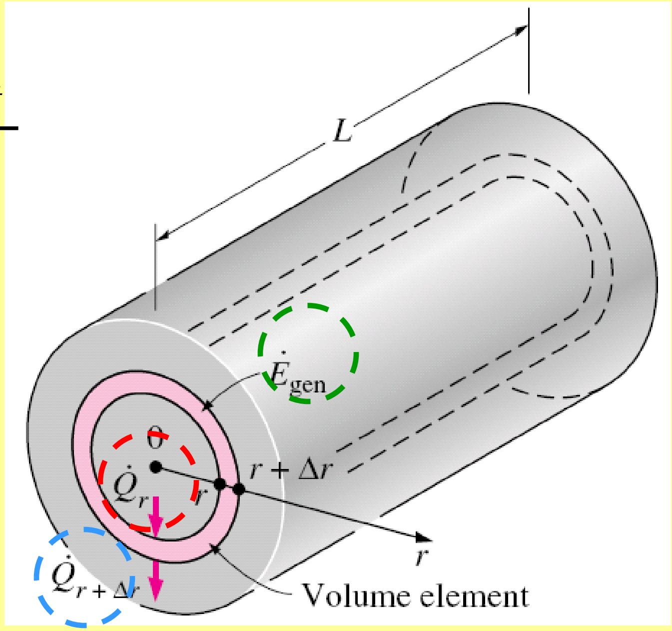
2) Transient, no heat generation:
$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation:
$$\frac{d^2 T}{dx^2}=0$$

One-Dimensional Heat Conduction Equation - Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta r \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r \end{cases}$$

- Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A\Delta r$, taking the limit as Δr→ 0 and Δt→ 0, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Noting that the area varies with the independent variable r according to $A=2\pi rL$, the one dimensional transient heat conduction equation in a long cylinder becomes

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

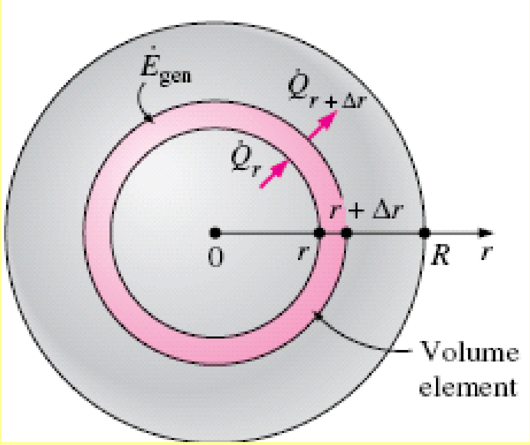
The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation: $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

One-Dimensional Heat Conduction Equation - Sphere



Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

EXAMPLE 2–4 Heat Conduction in a Resistance Heater

A 2-kW resistance heater wire with thermal conductivity $k = 15$ W/m · K, diameter $D = 0.4$ cm, and length $L = 50$ cm is used to boil water by immersing it in water (Fig. 2–19). Assuming the variation of the thermal conductivity of the wire with temperature to be negligible, obtain the differential equation that describes the variation of the temperature in the wire during steady operation.

SOLUTION The resistance wire of a water heater is considered. The differential equation for the variation of temperature in the wire is to be obtained.

Analysis The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial r direction only and thus the heat transfer to be one-dimensional. Then we have $T = T(r)$ during steady operation since the temperature in this case depends on r only.

The rate of heat generation in the wire per unit volume can be determined from

$$\dot{e}_{\text{gen}} = \frac{\dot{E}_{\text{gen}}}{V_{\text{wire}}} = \frac{\dot{E}_{\text{gen}}}{(\pi D^2/4)L} = \frac{2000 \text{ W}}{[\pi(0.004 \text{ m})^2/4](0.5 \text{ m})} = 0.318 \times 10^9 \text{ W/m}^3$$

Noting that the thermal conductivity is given to be constant, the differential equation that governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

which is the steady one-dimensional heat conduction equation in cylindrical coordinates for the case of constant thermal conductivity.

Discussion Note again that the conditions at the surface of the wire have no effect on the differential equation.

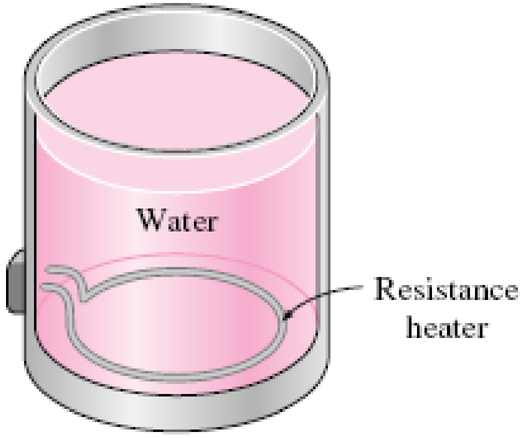
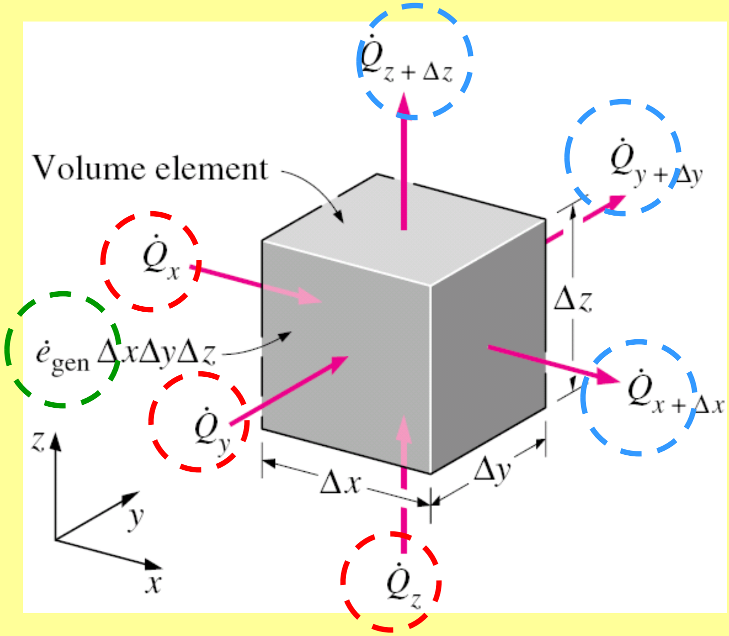


FIGURE 2–19
Schematic for Example 2–4.

General Heat Conduction Equation



Rate of heat conduction at *x*, *y*, and *z* **-** **Rate of heat conduction at *x+Δx*, *y+Δy*, *and z+Δz*** **+** **Rate of heat generation inside the element** **=** **Rate of change of the energy content of the element**

$$\overbrace{\dot{Q}_x + \dot{Q}_y + \dot{Q}_z} - \overbrace{\dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z}} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Constant conductivity:

$$\underbrace{\overbrace{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}}^{\textbf{Two-dimensional}}+\frac{\partial^2 T}{\partial z^2}}_{\textbf{Three-dimensional}}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1) Steady-state:

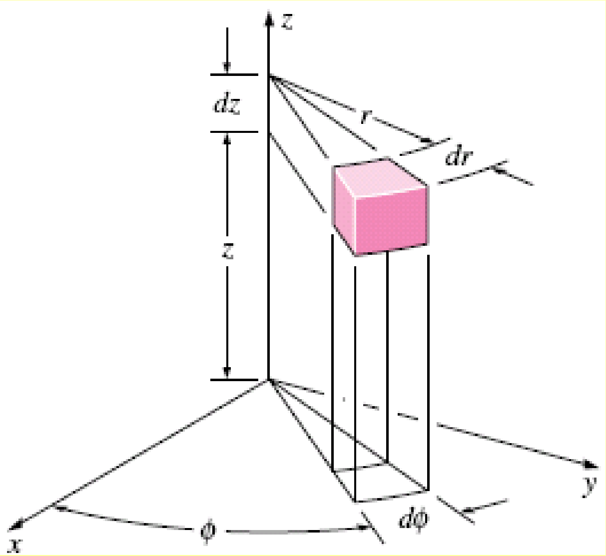
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

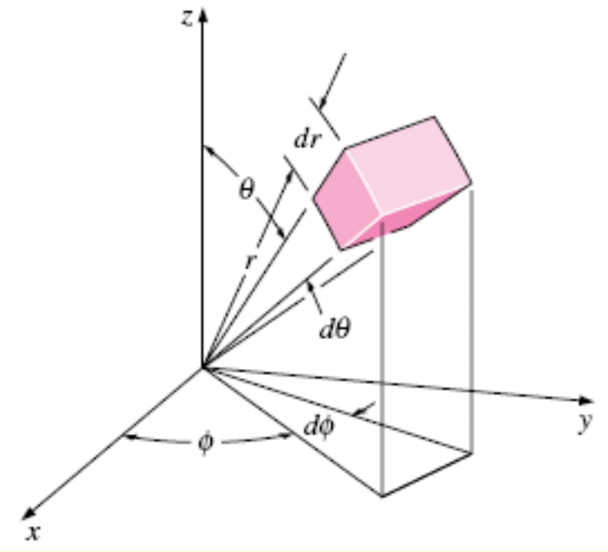
3) Steady-state, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$



Cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$



Spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

EXAMPLE 2–6 Heat Conduction in a Short Cylinder

A short cylindrical metal billet of radius R and height h is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 65$°F by convection and radiation. Assuming the billet is cooled uniformly from all outer surfaces and the variation of the thermal conductivity of the material with temperature is negligible, obtain the differential equation that describes the variation of the temperature in the billet during this cooling process.

SOLUTION A short cylindrical billet is cooled in ambient air. The differential equation for the variation of temperature is to be obtained.

Analysis The billet shown in Fig. 2–25 is initially at a uniform temperature and is cooled uniformly from the top and bottom surfaces in the z-direction as well as the lateral surface in the radial r-direction. Also, the temperature at any point in the ball changes with time during cooling. Therefore, this is a two-dimensional transient heat conduction problem since the temperature within the billet changes with the radial and axial distances r and z and with time t. That is, $T = T(r, z, t)$.

The thermal conductivity is given to be constant, and there is no heat generation in the billet. Therefore, the differential equation that governs the variation of temperature in the billet in this case is obtained from Eq. 2–43 by setting the heat generation term and the derivatives with respect to ϕ equal to zero. We obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho c\frac{\partial T}{\partial t}$$

In the case of constant thermal conductivity, it reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which is the desired equation.

Discussion Note that the boundary and initial conditions have no effect on the differential equation.

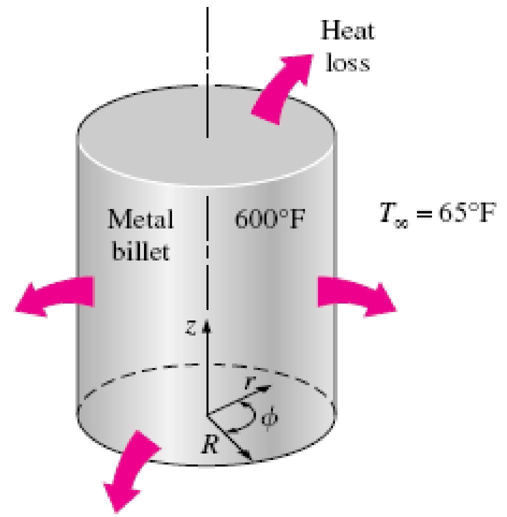


FIGURE 2–25
Schematic for Example 2–6.

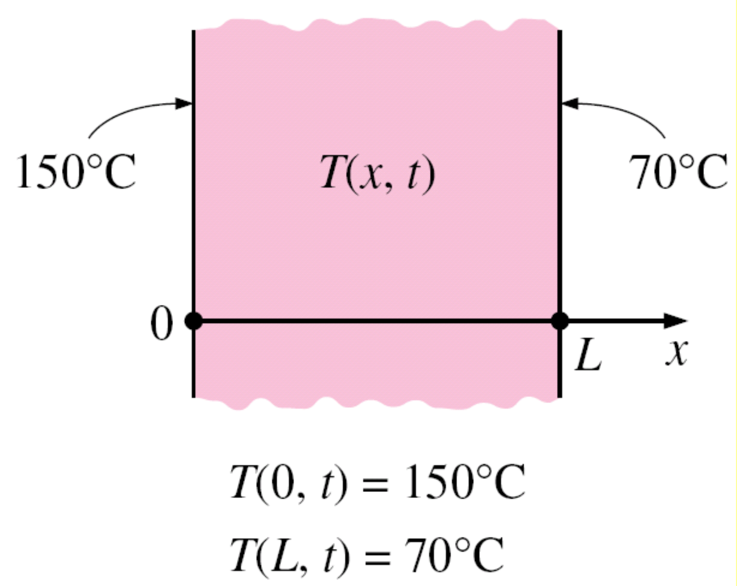
Boundary and Initial Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Specified Temperature Boundary Condition

For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary conditions can be expressed as

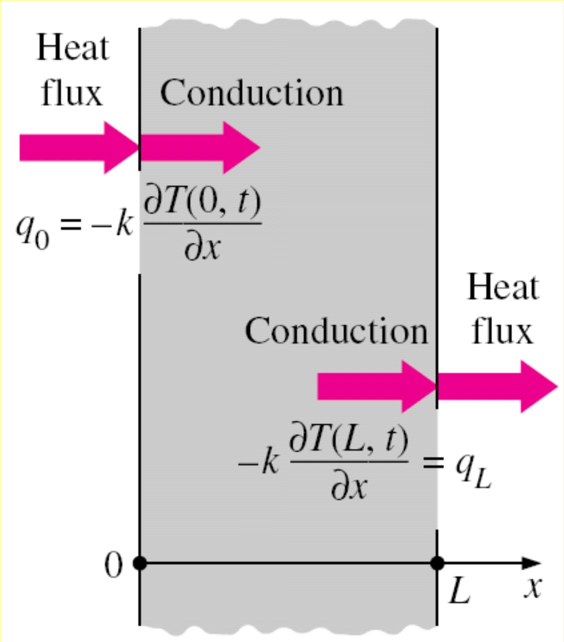
$$T(0, t) = T_1$$
$$T(L, t) = T_2$$



The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

The heat flux in the positive x-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as

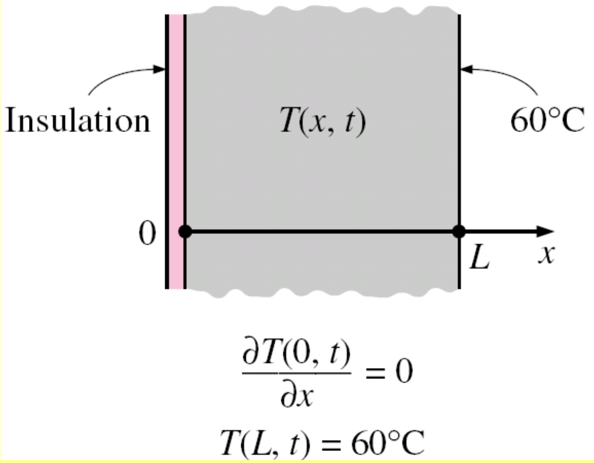


$$\dot{q} = -k\frac{dT}{dx} = \left(\text{Heat flux in the positive x-direction} \right)$$

The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

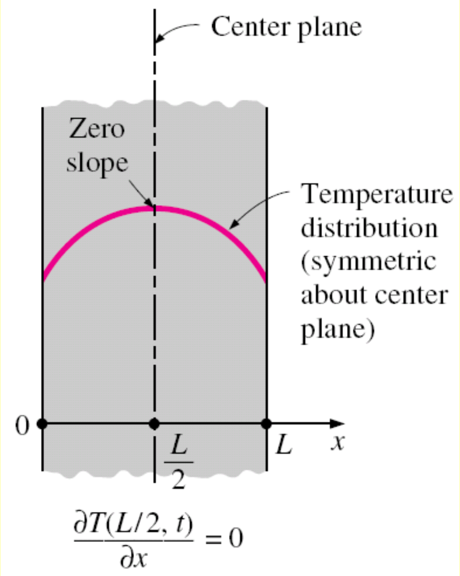
Two Special Cases

Insulated boundary



$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

Thermal symmetry



$$\frac{\partial T\left(L/2, t\right)}{\partial x} = 0$$

EXAMPLE 2–7 Heat Flux Boundary Condition

Consider an aluminum pan used to cook beef stew on top of an electric range. The bottom section of the pan is $L = 0.3$ cm thick and has a diameter of $D = 20$ cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 90 percent of the heat generated in the heating element is transferred to the pan. During steady operation, the temperature of the inner surface of the pan is measured to be 110°C. Express the boundary conditions for the bottom section of the pan during this cooking process.

SOLUTION An aluminum pan on an electric range top is considered. The boundary conditions for the bottom of the pan are to be obtained.

Analysis The heat transfer through the bottom section of the pan is from the bottom surface toward the top and can reasonably be approximated as being one-dimensional. We take the direction normal to the bottom surfaces of the pan as the x axis with the origin at the outer surface, as shown in Fig. 2–32. Then the inner and outer surfaces of the bottom section of the pan can be represented by $x = 0$ and $x = L$, respectively. During steady operation, the temperature will depend on x only and thus $T = T(x)$.

The boundary condition on the outer surface of the bottom of the pan at $x = 0$ can be approximated as being specified heat flux since it is stated that 90 percent of the 800 W (i.e., 720 W) is transferred to the pan at that surface. Therefore,

$$-k\frac{dT(0)}{dx} = \dot{q}_0$$

where

$$\dot{q}_0 = \frac{\text{Heat transfer rate}}{\text{Bottom surface area}} = \frac{0.720\text{ kW}}{\pi(0.1\text{ m})^2} = 22.9\text{ kW/m}^2$$

The temperature at the inner surface of the bottom of the pan is specified to be 110°C. Then the boundary condition on this surface can be expressed as

$$T(L) = 110°\text{C}$$

where $L = 0.003$ m.

Discussion Note that the determination of the boundary conditions may require some reasoning and approximations.

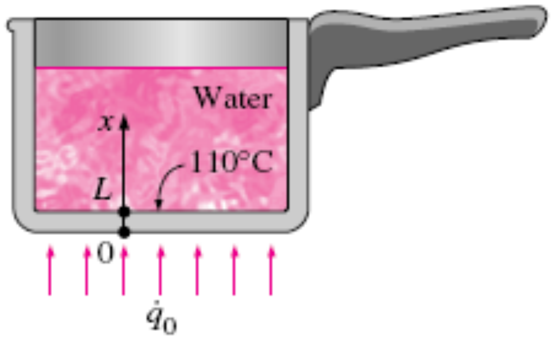


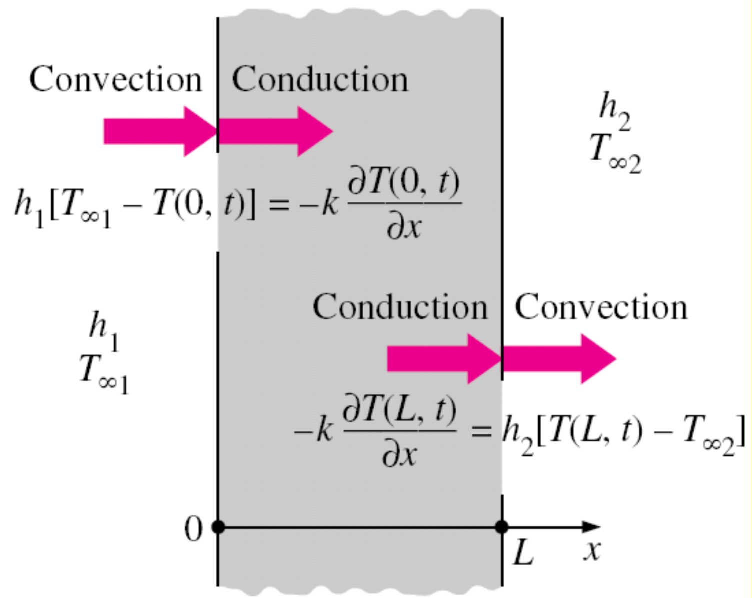
FIGURE 2–32
Schematic for Example 2–7.

Convection Boundary Condition

$$\left(\begin{array}{c}\text{Heat conduction} \\ \text{at the surface in} \\ \text{a selected} \\ \text{direction}\end{array}\right) = \left(\begin{array}{c}\text{Heat convection} \\ \text{at the surface in} \\ \text{the same} \\ \text{direction}\end{array}\right)$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1\left[T_{\infty 1} - T(0,t)\right]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2\left[T(L,t) - T_{\infty 2}\right]$$



EXAMPLE 2–8 Convection and Insulation Boundary Conditions

Steam flows through a pipe shown in Fig. 2–35 at an average temperature of T_∞ = 200°C. The inner and outer radii of the pipe are r_1 = 8 cm and r_2 = 8.5 cm, respectively, and the outer surface of the pipe is heavily insulated. If the convection heat transfer coefficient on the inner surface of the pipe is h = 65 W/m² · K, express the boundary conditions on the inner and outer surfaces of the pipe during transient periods.

SOLUTION The flow of steam through an insulated pipe is considered. The boundary conditions on the inner and outer surfaces of the pipe are to be obtained.

Analysis During initial transient periods, heat transfer through the pipe material predominantly is in the radial direction, and thus can be approximated as being one-dimensional. Then the temperature within the pipe material changes with the radial distance r and the time t. That is, $T = T(r, t)$.

It is stated that heat transfer between the steam and the pipe at the inner surface is by convection. Then taking the direction of heat transfer to be the positive r direction, the boundary condition on that surface can be expressed as

$$-k\frac{\partial T(r_1, t)}{\partial r} = h[T_\infty - T(r_1)]$$

The pipe is said to be well insulated on the outside, and thus heat loss through the outer surface of the pipe can be assumed to be negligible. Then the boundary condition at the outer surface can be expressed as

$$\frac{\partial T(r_2, t)}{\partial r} = 0$$

Discussion Note that the temperature gradient must be zero on the outer surface of the pipe at all times.

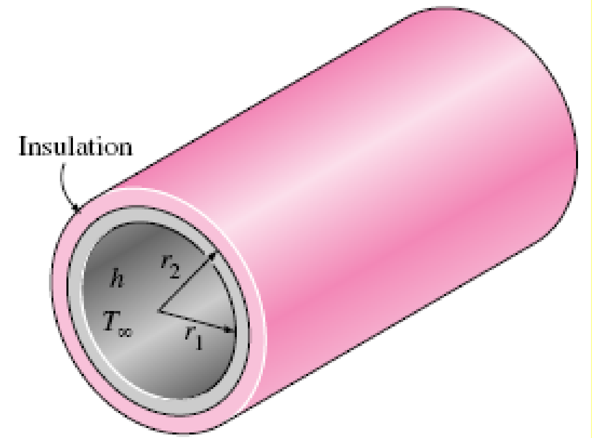


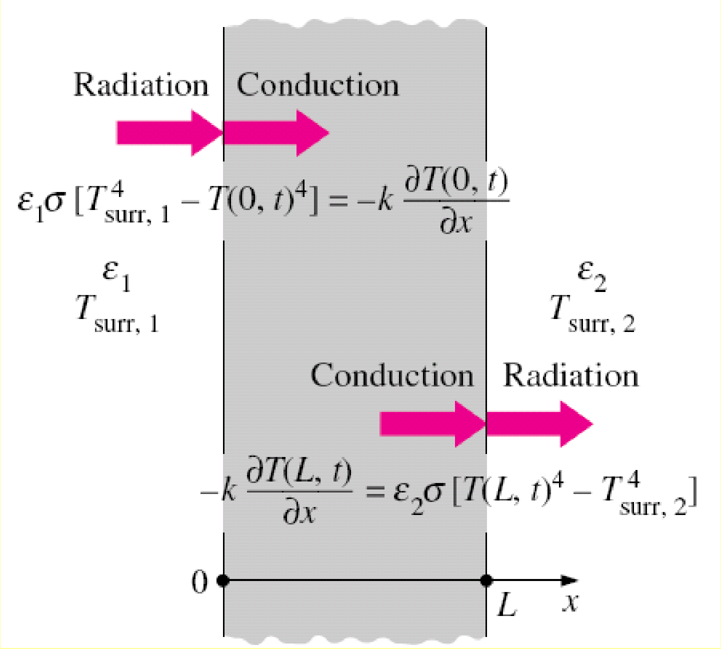
FIGURE 2–35
Schematic for Example 2–8.

Radiation Boundary Condition

(Heat conduction at the surface in a selected direction) = (Radiation exchange at the surface in the same direction)

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1 \sigma \left[T_{surr,1}^4 - T(0,t)^4 \right]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2 \sigma \left[T(L,t)^4 - T_{surr,2}^4 \right]$$



Interface Boundary Conditions

At the interface the requirements are:

(1) two bodies in contact must have the *same temperature* at the area of contact,

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0,t)}{\partial x} = -k_B \frac{\partial T_B(x_0,t)}{\partial x}$$

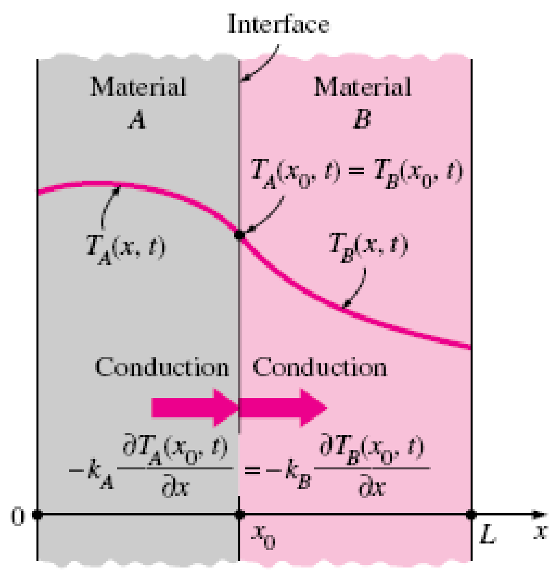


FIGURE 2–37
Boundary conditions at the interface of two bodies in perfect contact.

Generalized boundary condition

$$\left(\begin{array}{c} \text{Heat transfer} \\ \text{to the surface} \\ \text{in all modes} \end{array} \right) = \left(\begin{array}{c} \text{Heat transfer} \\ \text{from the surface} \\ \text{In all modes} \end{array} \right)$$

EXAMPLE 2–11 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m · °C, and surface area $A = 15$ m². The two sides of the wall are maintained at constant temperatures of $T_1 = 120°C$ and $T_2 = 50°C$, respectively, as shown in Fig. 2–41. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.
Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.
Properties The thermal conductivity is given to be $k = 1.2$ W/m · °C.
Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°C$$
$$T(L) = T_2 = 50°C$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1x + C_2$$

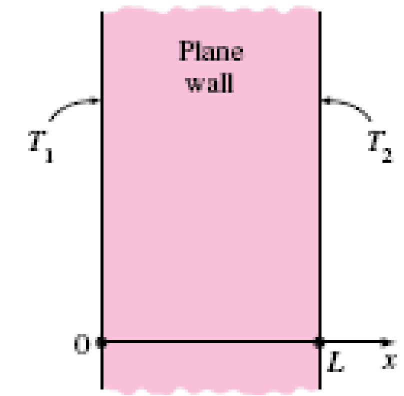


FIGURE 2–41
Schematic for Example 2–11.

Differential equation:

$$\frac{d^2T}{dx^2} = 0$$

Integrate:

$$\frac{dT}{dx} = C_1$$

Integrate again:

$$T(x) = C_1x + C_2$$

General solution

Arbitrary constants

FIGURE 2–42
Obtaining the general solution of a simple second order differential equation by integration.

which is the general solution of the differential equation (Fig. 2-42). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and T(x) by T_1*. That is (Fig. 2–43),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$\underbrace{\underset{\substack{\uparrow \\ 0}}{T(x)}}_{T_1} = C_1 \underset{\substack{\uparrow \\ 0}}{x} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–43
When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2*. That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \tag{2–56}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1\text{ m}) = \frac{(50 - 120)°\text{C}}{0.2\text{ m}}(0.1\text{ m}) + 120°\text{C} = 85°\text{C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \tag{2–57}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2\text{ W/m}\cdot°\text{C})(15\text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2\text{ m}} = 6300\text{ W}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

EXAMPLE 2–15 Heat Loss through a Steam Pipe

Consider a steam pipe of length $L = 20$ m, inner radius $r_1 = 6$ cm, outer radius $r_2 = 8$ cm, and thermal conductivity $k = 20$ W/m · °C, as shown in Fig. 2–50. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150°C$ and $T_2 = 60°C$, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

SOLUTION A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions 1 Heat transfer is steady since there is no change with time. 2 Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. 3 Thermal conductivity is constant. 4 There is no heat generation.

Properties The thermal conductivity is given to be $k = 20$ W/m · °C.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°C$$
$$T(r_2) = T_2 = 60°C$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

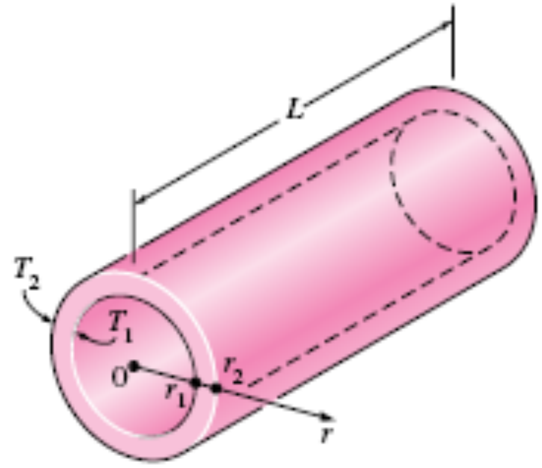


FIGURE 2–50
Schematic for Example 2–15.

Again integrating with respect to r gives (Fig. 2–51)

$$T(r) = C_1 \ln r + C_2 \qquad (a)$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in Eq. (a) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \qquad \text{and} \qquad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Substituting them into Eq. (a) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_2 - T_1) + T_1 \qquad (2\text{–}58)$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

$$\dot{Q}_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad (2\text{–}59)$$

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

$$\dot{Q} = 2\pi(20\ \text{W/m}\cdot{}^\circ\text{C})(20\ \text{m})\frac{(150 - 60)^\circ\text{C}}{\ln(0.08/0.06)} = 786\ \text{kW}$$

Discussion Note that the total rate of heat transfer through a pipe is constant, but the heat flux $\dot{q} = \dot{Q}/(2\pi rL)$ is not since it decreases in the direction of heat transfer with increasing radius.

Differential equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate:

$$r\frac{dT}{dr} = C_1$$

Divide by r ($r \neq 0$):

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Integrate again:

$$T(r) = C_1 \ln r + C_2$$

which is the general solution.

FIGURE 2–51
Basic steps involved in the solution of the steady one-dimensional heat conduction equation in cylindrical coordinates.

Heat Generation in Solids- The Surface Temperature

Rate of heat transfer from the solid = Rate of energy generation within the solid

$$\dot{Q} = \dot{e}_{gen} V \qquad \text{(W)}$$

$$\dot{Q} = hA_s \left(T_s - T_\infty\right) \qquad \text{(W)}$$

$$T_s = T_\infty + \frac{\dot{e}_{gen} V}{hA_s}$$

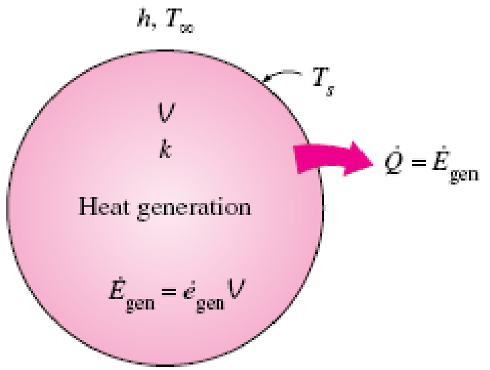


FIGURE 2–55
At steady conditions, the entire heat generated in a solid must leave the solid through its outer surface.

For a large *plane wall* of thickness $2L$ ($A_s=2A_{wall}$ and $V=2LA_{wall}$)

$$T_{s,plane\ wall} = T_\infty + \frac{\dot{e}_{gen} L}{h}$$

For a long solid *cylinder* of radius r_0 ($A_s=2\pi r_0 L$ and $V=\pi r_0^2 L$)

$$T_{s,cylinder} = T_\infty + \frac{\dot{e}_{gen} r_0}{2h}$$

For a solid *sphere* of radius r_0 ($A_s=4\pi r_0^2$ and $V={}^4/_3\pi r_0^3$)

$$T_{s,sphere} = T_\infty + \frac{\dot{e}_{gen} r_0}{3h}$$

Examples of heat generation

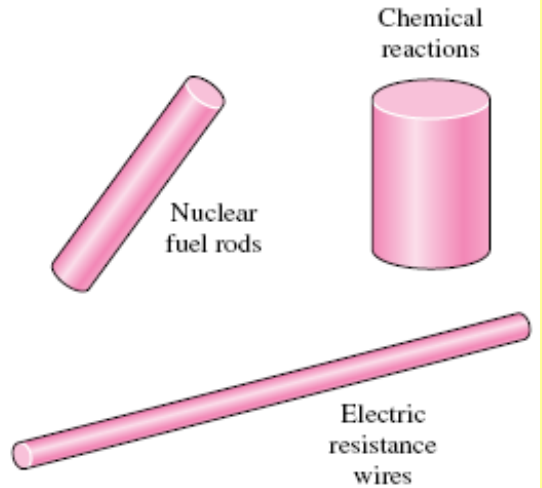


FIGURE 2–54
Heat generation in solids is commonly encountered in practice.

Heat Generation in Solids -The maximum Temperature in a Cylinder (the Centerline)

The *heat generated* within an inner cylinder must be equal to the *heat conducted* through its outer surface.

$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r$$

Substituting these expressions into the above equation and separating the variables, we get

$$-k\left(2\pi r L\right)\frac{dT}{dr} = \dot{e}_{gen}\left(\pi r^2 L\right) \rightarrow dT = -\frac{\dot{e}_{gen}}{2k} r dr$$

Integrating from $r = 0$ where $T(0) = T_0$ to $r = r_o$

Cylinder

$$\Delta T_{\text{max, cylinder}} = T_0 - T_s = \frac{\dot{e}_{\text{gen}} r_o^2}{4k}$$

Plane wall

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{\text{gen}} L^2}{2k}$$

Sphere

$$\Delta T_{\text{max, sphere}} = \frac{\dot{e}_{\text{gen}} r_o^2}{6k}$$

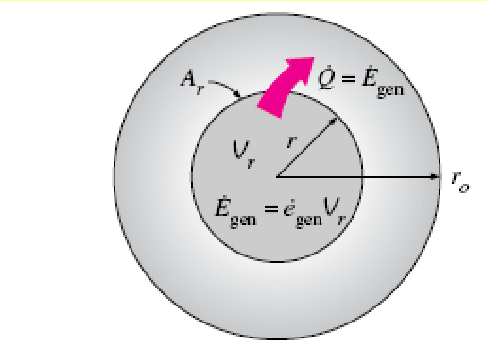


FIGURE 2–56
Heat conducted through a cylindrical shell of radius r is equal to the heat generated within a shell.

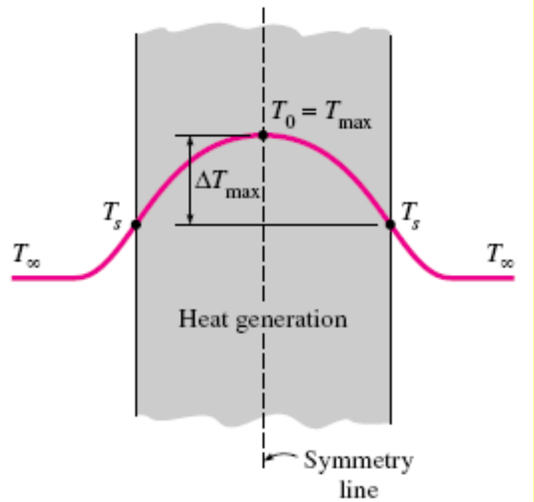


FIGURE 2–57
The maximum temperature in a symmetrical solid with uniform heat generation occurs at its center.

EXAMPLE 2–19 **Heat Conduction in a Two-Layer Medium**

Consider a long resistance wire of radius $r_1 = 0.2$ cm and thermal conductivity $k_{wire} = 15$ W/m · °C in which heat is generated uniformly as a result of resistance heating at a constant rate of $\dot{e}_{gen} = 50$ W/cm³ (Fig. 2–61). The wire is embedded in a 0.5-cm-thick layer of ceramic whose thermal conductivity is $k_{ceramic} = 1.2$ W/m · °C. If the outer surface temperature of the ceramic layer is measured to be $T_s = 45$°C, determine the temperatures at the center of the resistance wire and the interface of the wire and the ceramic layer under steady conditions.

SOLUTION The surface and interface temperatures of a resistance wire covered with a ceramic layer are to be determined.

Assumptions 1 Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since this two-layer heat transfer problem possesses symmetry about the centerline and involves no change in the axial direction, and thus $T = T(r)$. **3** Thermal conductivities are constant. **4** Heat generation in the wire is uniform.

Properties It is given that $k_{wire} = 15$ W/m · °C and $k_{ceramic} = 1.2$ W/m · ° C.

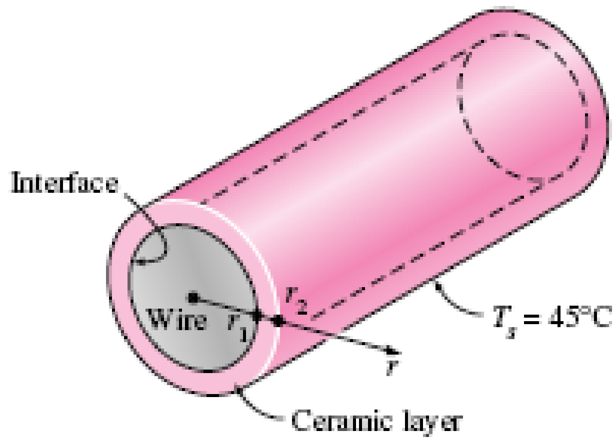


FIGURE 2–61
Schematic for Example 2–19.

Analysis Letting T_I denote the unknown interface temperature, the heat transfer problem in the wire can be formulated as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_{wire}}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

with

$$T_{wire}(r_1) = T_I$$

$$\frac{dT_{wire}(0)}{dr} = 0$$

This problem was solved in Example 2–18, and its solution was determined to be

$$T_{wire}(r) = T_I + \frac{\dot{e}_{gen}}{4k_{wire}}(r_1^2 - r^2) \qquad (a)$$

Noting that the ceramic layer does not involve any heat generation and its outer surface temperature is specified, the heat conduction problem in that layer can be expressed as

$$\frac{d}{dr}\left(r\frac{dT_{\text{ceramic}}}{dr}\right) = 0$$

with

$$T_{\text{ceramic}}(r_1) = T_I$$
$$T_{\text{ceramic}}(r_2) = T_s = 45°\text{C}$$

This problem was solved in Example 2–15, and its solution was determined to be

$$T_{\text{ceramic}}(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_s - T_I) + T_I \qquad (b)$$

We have already utilized the first interface condition by setting the wire and ceramic layer temperatures equal to T_I at the interface $r = r_1$. The interface temperature T_I is determined from the second interface condition that the heat flux in the wire and the ceramic layer at $r = r_1$ must be the same:

$$-k_{\text{wire}}\frac{dT_{\text{wire}}(r_1)}{dr} = -k_{\text{ceramic}}\frac{dT_{\text{ceramic}}(r_1)}{dr} \quad \rightarrow \quad \frac{\dot{e}_{\text{gen}}r_1}{2} = -k_{\text{ceramic}}\frac{T_s - T_I}{\ln(r_2/r_1)}\left(\frac{1}{r_1}\right)$$

Solving for T_I and substituting the given values, the interface temperature is determined to be

$$T_I = \frac{\dot{e}_{\text{gen}}r_1^2}{2k_{\text{ceramic}}}\ln\frac{r_2}{r_1} + T_s$$
$$= \frac{(50 \times 10^6\ \text{W/m}^3)(0.002\ \text{m})^2}{2(1.2\ \text{W/m}\cdot°\text{C})}\ln\frac{0.007\ \text{m}}{0.002\ \text{m}} + 45°\ \text{C} = 149.4°\text{C}$$

Knowing the interface temperature, the temperature at the centerline ($r = 0$) is obtained by substituting the known quantities into Eq. (*a*),

$$T_{\text{wire}}(0) = T_I + \frac{\dot{e}_{\text{gen}}r_1^2}{4k_{\text{wire}}} = 149.4°\text{C} + \frac{(50 \times 10^6\ \text{W/m}^3)(0.002\ \text{m})^2}{4 \times (15\ \text{W/m}\cdot°\text{C})} = 152.7°\text{C}$$

Variable Thermal Conductivity, $k(T)$

- The thermal conductivity of a material, in general, varies with temperature.
- An average value for the thermal conductivity is commonly used when the variation is mild.
- This is also common practice for other temperature-dependent properties such as the density and specific heat.

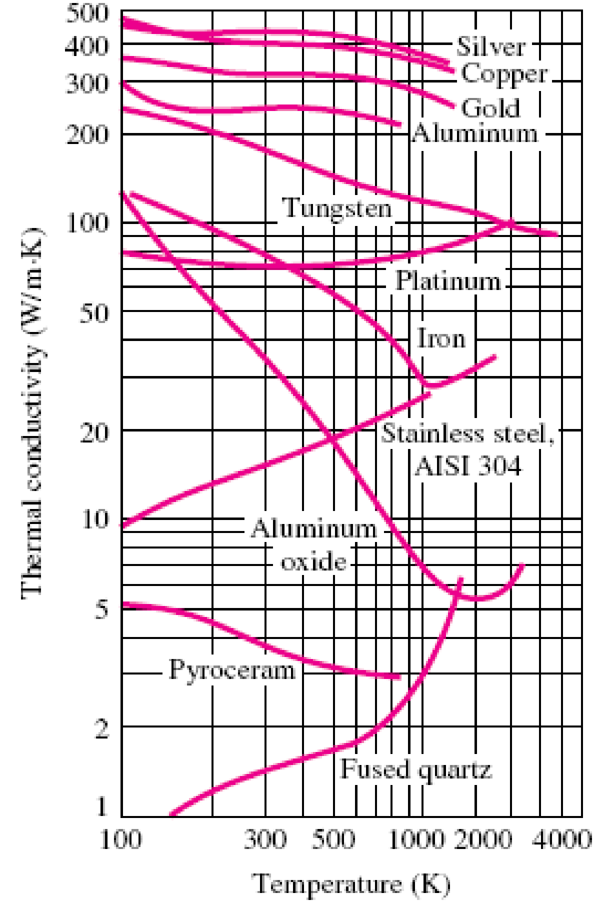


FIGURE 2–62
Variation of the thermal conductivity of some solids with temperature.

Variable Thermal Conductivity for One-Dimensional Cases

When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

The variation in thermal conductivity of a material with can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

β is the **temperature coefficient of thermal conductivity**.

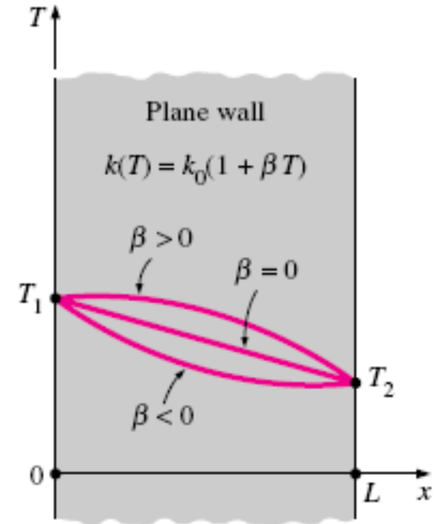


FIGURE 2–63
The variation of temperature in a plane wall during steady one-dimensional heat conduction for the cases of constant and variable thermal conductivity.

For a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant. This is no longer the case when the thermal conductivity changes with temperature (even linearly).

Concluding Points

- One-Dimensional Heat Conduction
- General Heat Conduction Equation
- Boundary and Initial Conditions
- Solution of Steady One-Dimensional Heat Conduction Problems
- Heat Generation in a Solid
- Variable Thermal Conductivity $k(T)$