Sabit Katsayılı Lineer Diferansiyel Denklemlerin Laplace Dönüşümü ile Çözümü

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(t) \qquad (y = y(t))$$

$$y_0 = y(0), \; y_1 = y'(0), \; y_2 = y''(0), \dots, \; y_{n-1} = y^{(n-1)}(0)$$

problemini gözönüne alalım.

Verilen diferansiyel denklemin her teriminin Laplace dönüşümü alınarak bilinmeyen çözüm fonksiyonunun dönüşüm cinsinden bir cebirsel denklemi elde edilir. Çözüm fonksiyonu, dönüşüm parametresi cinsinden düzenlenir ve ters Laplace dönüşümü alınarak çözüm fonksiyonuna ulaşılır.

$$\mathcal{L}\{a_0 y^{(n)}\} + \mathcal{L}\{a_1 y^{(n-1)}\} + \dots + \mathcal{L}\{a_{n-1} y'\} + \mathcal{L}\{a_n y\} = \mathcal{L}\{f(t)\}$$

$$a_0 \mathcal{L}\{y^{(n)}\} + a_1 \mathcal{L}\{y^{(n-1)}\} + \dots + a_{n-1} \mathcal{L}\{y'\} + a_n \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$a_0 \left[s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) + \dots\right] + a_1 \left[s^{n-1} Y(s) - s^{n-2} y(0) - s^{n-3} y'(0) + \dots\right] + \dots + a_{n-1}\left[sY(s) - y(0)\right] + a_n Y(s) = F(s)$$

$$\vdots$$

$$Y(s) = \text{-----}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{(\text{----})\} \Rightarrow y(t) = \text{----}$$

Ör/ $y'' - y' - 2y = 0$ $\quad y(0) = 1$

$y'(0) = 0$

a) Normal çözüm

b) Laplace çözüm

a) $y'' - y' - 2y = 0$

$(D^2 - D - 2)y = 0$

K.D: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \quad r_2 = -1$

$y = c_1 e^{2t} + c_2 e^{-t}$

$y' = 2c_1 e^{2t} - c_2 e^{-t}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow 2c_1 - c_2 = 0$

$c_1 = 1/3 \Rightarrow c_2 = \frac{2}{3}$

$\Rightarrow y = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$

b) $\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\} \Rightarrow s^2 Y(s) - s\,y(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0$

$\Rightarrow (s^2 - s - 2) Y(s) - s + 1 = 0$

$\Rightarrow Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)}$

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s+1)(s-2)}$$

$$\frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = \frac{2}{3} \qquad B = \frac{1}{3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)(s-2)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2/3}{s+1} + \frac{1/3}{s-2}\right\}$$

$$= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$

Ör/ $y'' + 4y = 0 \quad y(0) = 1,\ y'(0) = -2$ probleminin çözümünü Laplace dönüşümü ile bulunuz.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 0$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = 0$$

$$\Rightarrow s^2 Y(s) - s + 2 + 4Y(s) = 0 \Rightarrow (s^2+4)Y(s) = s-2$$

$$Y(s) = \frac{s-2}{s^2+4}$$

$$Y(s)=\frac{s-2}{s^2+4}=\frac{s}{s^2+4}-\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$=\cos 2t-\sin 2t$$

Ör/ $y''-6y'+9y=t^2\cdot e^{3t}$ $\quad y(0)=2,\ y'(0)=6$ probleminin çözümünü Laplace dönüşümü ile bulunuz.

$$\mathcal{L}\{y''-6y'+9y\}=\mathcal{L}\{t^2e^{3t}\}$$

$$\mathcal{L}\{y''\}-6\mathcal{L}\{y'\}+9\mathcal{L}\{y\}=\frac{2}{(s-3)^3}$$

$$s^2Y(s)-sy(0)-y'(0)-6(sY(s)-y(0))+9Y(s)=\frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s)-2s-6+12=\frac{2}{(s-3)^3}$$

$$(s-3)^2Y(s)-2(s-3)=\frac{2}{(s-3)^3}\Rightarrow(s-3)^2Y(s)=\frac{2}{(s-3)^3}+2(s-3)$$

$$Y(s)=\frac{2}{(s-3)^5}+\frac{2}{s-3}$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2 \cdot 4!}{(s-3)^5 \cdot 4!}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}$$

$$y(t) = \frac{2}{4!} \cdot t^4 \cdot e^{3t} + 2e^{3t} = \frac{1}{12} t^4 e^{3t} + 2e^{3t}$$

Ör/ $y'' + 4y' + 4y = 5\cos 2t \quad y(0)=0, \; y'(0)=0$ probleminin çözümünü Laplace dönüşümü ile bulunuz.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{5\cos 2t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = 5 \cdot \frac{s}{s^2+4}$$

$$\underbrace{(s^2+4s+4)}_{(s+2)^2} Y(s) = \frac{5s}{s^2+4} \Rightarrow Y(s) = \frac{5s}{(s+2)^2(s^2+4)} \Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5s}{(s+2)^2(s^2+4)}\right\}$$

$$\frac{5s}{(s+2)^2(s^2+4)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+4}$$

$$5s \equiv A(s+2)(s^2+4) + B(s^2+4) + (Cs+D)(s+2)^2$$

$$5s \equiv A(s+2)(s^2+4) + B(s^2+4) + (Cs+D)(s+2)^2$$

$$5s \equiv A(s^3+4s+2s^2+8) + B(s^2+4) + (Cs+D)(s^2+4s+4)$$

$$5s \equiv As^3+4As+2As^2+8A+Bs^2+4B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D$$

$$5s \equiv (A+C)s^3+(2A+B+4C+D)s^2+(4A+4C+4D)s+(8A+4B+4D)$$

$$A+C=0 \longrightarrow A=-C$$

$$\begin{cases} 2A+B+4C+D=0 \\ 4A+4C+4D=5 \\ 8A+4B+4D=0 \end{cases} \Rightarrow \quad 4D=5 \Rightarrow D=\frac{5}{4}$$

$$2A+B-4A+D=0 \xRightarrow{4/} -2A+B=\frac{-5}{4}$$

$$8A+4B=-5$$

$$8B=-10$$

$$B=\frac{-10}{8}=\frac{-5}{4} \Rightarrow A=0 \Rightarrow C=0$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5s}{(s+2)^2(s^2+4)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-5/4}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-5/4}{s^2+4}\right\}$$

$$= -\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} - \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{2\cdot 1}{2(s^2+4)}\right\} = -\frac{5}{4}t.e^{-2t} - \frac{5}{4}\cdot\frac{1}{2}\sin 2t = \frac{-5}{4}te^{-2t} - \frac{5}{8}\sin 2t$$

Ödev $k \neq 0$ bir reel sabit olmak üzere $y'' + k^2 y = t$ $y(0) = 1$, $y'(0) = -2$ probleminin çözümünü Laplace dönüşümü ile bulunuz.

Örnek $y''' - y'' + y' - y = 0$ $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ başlangıç değer probleminin çözümünü Laplace dönüşümü ile bulunuz.

$$\mathcal{L}\{y''' - y'' + y' - y\} = 0$$

$$\mathcal{L}\{y'''\} - \mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{y\} = 0$$

$$s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - \left(s^2 Y(s) - s y(0) - y'(0)\right) + \left(s Y(s) - y(0)\right) - Y(s) = 0$$

$$\left(s^3 - s^2 + s - 1\right) Y(s) - s^2 + s - 1 = 0$$

$$\Rightarrow \quad Y(s) = \frac{s^2 - s + 1}{s^3 - s^2 + s - 1} = \frac{s^2 - s + 1}{s^2(s-1) + (s-1)} = \frac{s^2 - s + 1}{(s-1)(s^2+1)}$$

$$\frac{s^2 - s + 1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs + C}{s^2+1} \Rightarrow \begin{array}{l} s^2 - s + 1 \equiv A(s^2+1) + (Bs+C)(s-1) \\ s^2 - s + 1 \equiv As^2 + A + Bs^2 - Bs + Cs - C \end{array}$$

$$s^2 - s + 1 \equiv As^2 + A + Bs^2 - Bs + Cs - C$$

$$s^2 - s + 1 \equiv (A+B)s^2 + (C-B)s + A - C$$

$$\left.\begin{aligned} A + B &= 1 \\ C - B &= -1 \end{aligned}\right\}$$

$$A - C = 1$$

$$\begin{aligned} A + C &= 0 \\ A - C &= 1 \\ \hline 2A &= 1 \end{aligned}$$

$$A = 1/2 \Rightarrow C = -\frac{1}{2}$$

$$B = 1/2$$

$$Y(s) = \frac{1/2}{s-1} + \frac{1/2\, s - 1/2}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = 1/2\, \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 1/2\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$$