$\tan \delta_n \pi - \frac{c_1}{n} + O\left(\frac{1}{n^2}\right) = 0$

$$\tan \delta_n \pi = \delta_n \pi + O\left[(\delta_n \pi)^3\right]$$

$$\delta_n = O\left(\frac{1}{n}\right) \Rightarrow (\delta_n \pi)^3 = O\left(\frac{1}{n^3}\right) = O\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \tan \delta_n \pi = \delta_n \pi + O\left(\frac{1}{n^2}\right)$$

ve 'dan

$$\delta_n \pi + O\left(\frac{1}{n^2}\right) - \frac{c_1}{n} + O\left(\frac{1}{n^2}\right) = 0$$

$\lim_{n\to\infty} \delta_n = 0 \quad |\delta_n| \le \frac{c}{n}$

$\Downarrow$

$\lim_{n\to\infty} \delta_n \pi = 0$

$\Downarrow$

$\exists n \in \mathbb{N}$, $n \ge N$ iken $|\delta_n \pi| < \varepsilon$ 'dur.

$\delta_n = O\left(\frac{1}{n}\right) \Rightarrow |\delta_n| \le \frac{c}{n}$

$|(\delta_n \pi)^3| = \pi^3 \cdot |\delta_n|^3 \le \pi^3 \cdot \frac{c^3}{n^3} \le \frac{c^3 \cdot \pi^3}{n^2}$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$$\begin{array}{l|l} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \hline -x + \frac{x^3}{2!} \mp \frac{x^5}{4!} - \cdots & x + \frac{x^3}{3} + \cdots \\ \hline \frac{x^3}{3} - \frac{4x^5}{5!} + \cdots & \\ -\frac{x^3}{3} + \frac{x^5}{3 \cdot 2} + \cdots & \\ \hline \end{array}$$

Öyle $\varepsilon > 0$ ve $A > 0$ sabitleri vardır ki $|x| < \varepsilon$ için

$x \to 0 \quad \tan x = x + O(x^3) \Rightarrow |\tan x - x| = |O(x^3)| \le Ax^3$ $(x \ge 0)$

$\star \quad \tan \delta_n \pi - \frac{c_1}{n} + O\left(\frac{1}{n^2}\right) = 0$

$$\tan \delta_n \pi = \delta_n \pi + O\left[(\delta_n \pi)^3\right]$$

$$\delta_n = O\left(\frac{1}{n}\right) \Rightarrow (\delta_n \pi)^3 = O\left(\frac{1}{n^3}\right) = O\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \tan \delta_n \pi = \delta_n \pi + O\left(\frac{1}{n^2}\right) \quad \star\star$$

$\lim_{n\to\infty} \delta_n = 0 \quad |\delta_n| \le \frac{c}{n}$

$\Downarrow$

$\lim_{n\to\infty} \delta_n \pi = 0$

$\Downarrow$

$\exists n \in \mathbb{N}$; $n \ge N$ iken $|\delta_n \pi| < \varepsilon$'dur.

$\delta_n = O\left(\frac{1}{n}\right) \Rightarrow |\delta_n| \le \frac{c}{n}$

$$|(\delta_n \pi)^3| = \pi^3 \cdot |\delta_n|^3 \le \pi^3 \cdot \frac{c^3}{n^3} \le \frac{c^3 \cdot \pi^3}{n^2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$$\begin{array}{l|l} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \hline -\cancel{x} + \frac{x^3}{2!} \mp \frac{x^5}{4!} - \cdots & x + \frac{x^3}{3} + \cdots \\ \hline \frac{x^3}{3} - \frac{4x^5}{5!} + \cdots & \\ -\cancel{\frac{x^3}{3}} + \frac{x^5}{3.2} + \cdots & \end{array}$$

öyle $\varepsilon > 0$ ve $A > 0$ sabitleri vardır ki $|x| < \varepsilon$ için

$x \to 0 \quad \tan x = x + O(x^3) \Rightarrow |\tan x - x| = |O(x^3)| \le A x^3$

$(x \ge 0)$

$\star$ ve $\star\star$'dan

$$\delta_n \pi + O\left(\frac{1}{n^2}\right) - \frac{c_1}{n} + O\left(\frac{1}{n^2}\right) = 0$$

$$\delta_n \pi = \frac{c_1}{n} + O\left(\frac{1}{n^2}\right) \Rightarrow \delta_n = \frac{c_1}{n\pi} + O\left(\frac{1}{n^2}\right)$$

$c_1 = \left[H + h + \frac{1}{2}\int_0^\pi q(x)\,dx\right] \qquad \frac{c_1}{\pi} = c$ olsun.

$$s_n = n + \delta_n \Rightarrow s_n = n + \frac{c}{n} + O\left(\frac{1}{n^2}\right)$$

$$\lambda_n = s_n^2 = \left[n + \frac{c}{n} + O\left(\frac{1}{n^2}\right)\right]^2 = n^2 + \frac{c^2}{n^2} + O\left(\frac{1}{n^4}\right) + 2c + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^3}\right)$$

$\boxed{\lambda_n = n^2 + 2c + O\left(\frac{1}{n}\right)}$ elde ederiz.

$q(x)$ fonksiyonunun $[0,\pi]$ aralığında 2. mertebeden sürekli türeve sahip olduğu kabul edilirse

$$s_n = n + \frac{c}{n} + \frac{c_1}{n^3} + O\left(\frac{1}{n^4}\right)$$ formülü ispatlanabilir.

Bu formülden

$$\lambda_n = s_n^2 = \left[n + \frac{c}{n} + \frac{c_1}{n^3} + O\left(\frac{1}{n^4}\right)\right]^2 = n^2 + \frac{c^2}{n^2} + \frac{c_1^2}{n^6} + 2c + 2\frac{c_1}{n^2} + O\left(\frac{1}{n^3}\right)$$

$$= n^2 + 2c + \frac{c_2}{n^2} + O\left(\frac{1}{n^3}\right)$$

$$\boxed{\lambda_n = n^2 + c_0 + \frac{c_2}{n^2} + O\left(\frac{1}{n^3}\right)}$$

$$c^2 + 2c_1 = c_2$$

$$c_0 = 2c = 2\frac{c_1}{\pi}$$

$$= \frac{2}{\pi}\left[H + h + \frac{1}{2}\int_0^{\pi} q(x)\,dx\right]$$

Problemimizin özfonksiyonları için bir formül bulalım;

$$\varphi(x,\lambda_n) = \varphi_n(x)$$

$$\varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau)\,\varphi(\tau,\lambda)\,d\tau$$

$$\varphi(x,\lambda) = \cos sx + O\left(\frac{1}{s}\right) \qquad (s \geq s_0 > 0)$$

$$\Rightarrow \varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x q(\tau)\cdot\left[\sin sx\cos s\tau - \cos sx\sin s\tau\right]\cdot\left[\cos s\tau + O\left(\frac{1}{s}\right)\right]d\tau$$

$$\varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x q(z).\left[\sin sx \cos sz - \cos sx \sin sz\right].\left[\cos sz + O\left(\frac{1}{s}\right)\right] dz$$

$$= \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x q(z)\left[\sin sx \cos^2 sz - \cos sx \sin sz \cos z\right] dz$$

$$+ \frac{1}{s}\int_0^x q(z)\sin sx \cos sz \, O\left(\frac{1}{s}\right) dz - \frac{1}{s}\int_0^x q(z)\cos sx \sin sz \, O\left(\frac{1}{s}\right) dz$$

$$= \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x q(z)\left[\sin sx\left(\frac{1+\cos 2sz}{2}\right) - \cos sx . \frac{\sin 2sz}{2}\right] dz$$

$$+ \frac{1}{s}\int_0^x q(z)\sin sx \cos sz \; O\left(\frac{1}{s}\right) dz - \frac{1}{s}\int_0^x q(z)\cos sx \sin sz \, O\left(\frac{1}{s}\right) dz$$

$$= \cos sx + \frac{h}{s}\sin sx + \frac{1}{2s}\int_0^x q(z)\left[\sin sx\,(1+\cos 2sz) - \cos sx\,.\sin 2sz\right] dz$$

$$+ \frac{1}{s}\int_0^x q(z)\sin sx \cos sz \; O\left(\frac{1}{s}\right) dz - \frac{1}{s}\int_0^x q(z)\cos sx \sin sz \, O\left(\frac{1}{s}\right) dz$$

- $$\left|\int_0^x q(z).\cos 2sz \, dz\right| = \left|\int_0^x q(z).\frac{1}{2s}.(\sin 2sz)' dz\right| = \left|\frac{1}{2s}q(z)\sin 2sz\Big|_0^x - \int_0^x \frac{1}{2s}q'(z).\sin 2sz)dz\right|$$

$q(z)=u \quad \frac{1}{2s}(\sin 2sz)' dz = dv$

$q'(z)dz = du \quad \frac{1}{2s}\sin 2sz = v$

$|a-b| \leq |a| + |b|$

$$\left| \frac{1}{2s} q(z)\sin 2sz \Big|_0^x - \int_0^x \frac{1}{2s} q'(z).\sin 2sz)\,dz \right| \le \left| \frac{1}{2s} q(z)\sin 2sz \Big|_0^x \right| + \left| \frac{1}{2s} \int_0^x q'(z)\sin 2sz\,dz \right|$$

$$\le \frac{1}{2s} \max_{0\le z\le\pi} |q(z)| + \frac{\pi}{2s} \max_{0\le z\le\pi} |q'(z)| \le \frac{c}{s}$$

- $\left| \int_0^x q(z)\,.\sin 2sz\,dz \right| \le \frac{c_1}{s}$ (Benzer şekilde elde edilir)
- $\left| \int_0^x q(z)\cos sz\, O\left(\frac{1}{s}\right) dz \right| \le \int_0^x \left|O\left(\frac{1}{s}\right)\right| |q(z)|\,dz \le \frac{c_2}{s} \int_0^x |q(z)|\,dz \le \frac{c_2}{s} \max_{0\le z\le\pi} |q(z)| \le c_3.\frac{1}{s}$
- $\left| \int_0^x q(z)\sin sz\, O\left(\frac{1}{s}\right) dz \right| \le \frac{c_4}{s}$ (Benzer şekilde elde edilir)

$$\boxed{\varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{\sin sx}{2s}\int_0^x q(z)\,dz + O\left(\frac{1}{s^2}\right)}$$

$$\varphi_n(x) = \varphi(x,\lambda_n) = \cos s_n x + \frac{h}{s_n}\sin s_n x + \frac{\sin s_n x}{2s_n}\int_0^x q(z)\,dz + O\left(\frac{1}{s_n^2}\right)$$

$$s_n = n + \frac{c}{n} + O\left(\frac{1}{n^2}\right) = n + O\left(\frac{1}{n}\right)$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$$

• $\cos s_n x = \cos\left[n+O\left(\frac{1}{n}\right)\right]x = \cos nx \cdot \cos\left[O\left(\frac{1}{n}\right)\right]x - \sin nx \cdot \sin\left[O\left(\frac{1}{n}\right)\right]x$

$$\left.\begin{aligned} \cos x &= 1+O(x^2) \\ \sin x &= x+O(x^3) \\ &= x+O(x^2) \end{aligned}\right\} \text{Taylor formülünden} \quad x\in[0,\pi]$$

$$\cos s_n x = \cos nx\left[1+O\left(\frac{1}{n^2}\right)\right] - \sin nx\left[O\left(\frac{1}{n}\right)x + O\left(\frac{1}{n^2}\right)\right]$$
$$= \cos nx + O\left(\frac{1}{n^2}\right) - \sin nx\left[\frac{c}{n}+O\left(\frac{1}{n^2}\right)\right]x - O\left(\frac{1}{n^2}\right)$$
$$= \cos nx - \frac{cx}{n}\cdot\sin nx + O\left(\frac{1}{n^2}\right)$$

• $\sin s_n x = \sin\left[n+O\left(\frac{1}{n}\right)x\right] = \sin nx \cdot \cos\left[O\left(\frac{1}{n}\right)\right]x + \cos nx \cdot \sin\left[O\left(\frac{1}{n}\right)\right]x$

$$= \sin nx\left[1+O\left(\frac{1}{n^2}\right)\right] + \cos nx\left[O\left(\frac{1}{n}\right)x + O\left(\frac{1}{n^2}\right)\right]$$
$$= \sin nx + O\left(\frac{1}{n^2}\right) + \cos nx\left[\frac{c}{n}+O\left(\frac{1}{n^2}\right)\right]x + O\left(\frac{1}{n^2}\right)$$
$$= \sin nx + \frac{cx}{n}\cos nx + O\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \varphi_n(x) = \cos nx - \frac{cx}{n}\sin nx + O\left(\frac{1}{n^2}\right) + \frac{h}{s_n}\left[\sin nx + \frac{cx}{n}\cos nx + O\left(\frac{1}{n^2}\right)\right] + \frac{\left[\sin nx + \frac{cx}{n}\cos nx + O\left(\frac{1}{n^2}\right)\right]}{2s_n}\int_0^x q(z)\,dz + O\left(\frac{1}{s_n^2}\right)$$

$$s_n = n + O\left(\frac{1}{n}\right)$$

$$\Rightarrow \frac{1}{s_n} = \frac{1}{n+O(\frac{1}{n})} = \frac{1}{n+O(\frac{1}{n})} + \frac{1}{n} - \frac{1}{n} = \frac{1}{n} + \left[\frac{1}{n+O(\frac{1}{n})} - \frac{1}{n}\right]$$

$$= \frac{1}{n} + \left[\frac{\cancel{n} - (\cancel{n} + O(\frac{1}{n}))}{n\,[n+O(\frac{1}{n})]}\right] = \frac{1}{n} + \left[\frac{O(\frac{1}{n})}{\underbrace{n(n+O(\frac{1}{n}))}}\right]$$

$$= \frac{1}{n} + \frac{O(\frac{1}{n})}{n^2} = \frac{1}{n} + O\left(\frac{1}{n^3}\right)$$

$$\Rightarrow \varphi_n(x) = \cos nx - \frac{cx}{n}\sin nx + \left[\frac{1}{n} + O\left(\frac{1}{n^3}\right)\right]\left[h + \frac{1}{2}\int_0^x q(z)\,dz\right].\left[\sin nx + \frac{cx}{n}\cos nx\right] + O\left(\frac{1}{n^2}\right)$$

$$\underbrace{\qquad}_{\{\; O(\frac{1}{n^3}).\sin nx = O(\frac{1}{n^2})}$$

$$= \cos nx - \frac{cx}{n}\sin nx + \left[h + \frac{1}{2}\int_0^x q(z)\,dz\right]\cdot\frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)$$

$$= \cos nx + \underbrace{\left[h + \frac{1}{2}\int_0^x q(z)\,dz - cx\right]}_{\beta(x)}\cdot\frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)$$

$$\boxed{\varphi_n(x) = \cos nx + \beta(x).\frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)}$$ formülü elde edilir.

$$\ell(y) = -y'' + q(x)\,y = \lambda y \qquad (3.1)$$

$$\left.\begin{aligned} &(a)\ \ y'(0) - h\,y(0) = 0 \\ &(b)\ \ y'(\pi) + H\,y(\pi) = 0 \end{aligned}\right\} \qquad (3.3)$$

problemi lineer olduğundan her $a \neq 0$ sayısı için $a.\varphi_n(x)$ de bu problemin λ_n özdeğerine karşı gelen özfonksiyonu olacaktır. Eğer özfonksiyonlar

$$\|\varphi_n\|^2 = \int_0^\pi [\varphi_n(x)]^2\,dx = 1 \qquad (n = 1, 2, \ldots)$$

koşulunu sağlıyorsa $\{\varphi_n(x)\}_1^\infty$'ye problemin normalleştirilmiş özfonksiyonları denir.

$$\frac{\varphi_n(x)}{\|\varphi_n(x)\|} = V_n(x) \rightarrow \text{problemin normalleştirilmiş özfonksiyonları olacaktır.}$$

$$\boxed{\varphi_n(x) = \cos nx + \beta(x).\frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)} \Rightarrow \|\varphi_n\|^2 = \int_0^\pi \left[\cos nx + \beta(x)\frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)\right]^2 dx$$

$$= \int_0^\pi \left[\cos^2 nx + \beta(x).\frac{\sin 2nx}{n} + O\left(\frac{1}{n^2}\right)\right] dx$$

$$= \int_0^\pi \left(\frac{1+\cos 2nx}{2}\right) dx - \int_0^\pi \beta(x).\frac{1}{2n^2}(\cos 2nx)'\,dx + O\left(\frac{1}{n^2}\right)$$

$$= \int_0^{\pi} \left(\frac{1+\cos 2nx}{2}\right) dx - \int_0^{\pi} \beta(x) \cdot \frac{1}{2n^2} (\cos 2nx)' dx + O\left(\frac{1}{n^2}\right)$$

$\beta(x) = u \Rightarrow \beta'(x) dx = du$

$\int (\cos 2nx)' dx = \int dv$

$\cos 2nx = v$

$$= \frac{1}{2}\int_0^{\pi} dx + \frac{1}{2}\int_0^{\pi} \cos 2nx\, dx - \frac{1}{2n^2}\left[\beta(x) \cdot \cos 2nx \Big|_0^{\pi} - \int_0^{\pi} \beta'(x) \cdot \cos 2nx\, dx\right]$$

$$+ O\left(\frac{1}{n^2}\right)$$

$$= \frac{\pi}{2} + \cancel{\frac{1}{2} \cdot \frac{1}{2n} \sin 2nx \Big|_0^{\pi}}^{\,0} + O\left(\frac{1}{n^2}\right)$$

$$= \frac{\pi}{2} + O\left(\frac{1}{n^2}\right) \Rightarrow \|\varphi_n\|^2 = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right) \Rightarrow \|\varphi_n\| = \sqrt{\frac{\pi}{2}} + O\left(\frac{1}{n^2}\right)$$

$$\frac{1}{\|\varphi_n\|} = \frac{1}{\sqrt{\frac{\pi}{2}} + O\left(\frac{1}{n^2}\right)} = \frac{1}{\sqrt{\frac{\pi}{2}}} - \frac{1}{\sqrt{\frac{\pi}{2}}} + \frac{1}{\sqrt{\frac{\pi}{2}} + O\left(\frac{1}{n^2}\right)} = \frac{1}{\sqrt{\frac{\pi}{2}}} + \left[\frac{1}{\sqrt{\frac{\pi}{2}} + O\left(\frac{1}{n^2}\right)} - \frac{1}{\sqrt{\frac{\pi}{2}}}\right]$$

$$= \sqrt{\frac{2}{\pi}} + \frac{O\left(\frac{1}{n^2}\right)}{\sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{\pi}{2}} + O\left(\frac{1}{n^2}\right)\right]} = \sqrt{\frac{2}{\pi}} + O\left(\frac{1}{n^2}\right)$$

$$v_n(x) = \frac{\varphi_n(x)}{\|\varphi_n(x)\|} = \left[\sqrt{\frac{2}{\pi}} + O\left(\frac{1}{n^2}\right)\right] \cdot \left[\cos nx + \beta(x) \cdot \frac{\sin nx}{n} + O\left(\frac{1}{n^2}\right)\right] = \sqrt{\frac{2}{\pi}}\left[\cos nx + \beta(x)\frac{\sin nx}{n}\right] + O\left(\frac{1}{n^2}\right)$$

şeklinde ortonormalleştirilmiş özfonksiyonlar için asimptotik formül elde edilir.