– Uygulama –

1) $f(x,y) = \dfrac{x^2+y^2-x^3y^3}{x^2+y^2}$ $\quad (x,y) \neq (0,0)$

fonksiyonu orjinde nasıl tanımlanmalıdır ki xy-düzleminin her noktasında sürekli olsun.

$$f(x,y) = \frac{x^2+y^2-x^3y^3}{x^2+y^2} = 1 - \frac{x^3y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=kx}} \frac{x^3\cdot(kx)^3}{x^2+(kx)^2}$$

$$= \lim_{\substack{x\to 0\\ y=kx}} \frac{k^3x^6}{x^2(1+k^2)} = \lim_{\substack{x\to 0\\ y=kx}} x^4 \cdot \frac{k^3}{1+k^2}$$

$$=0$$

dır. Hangi yol ile yaklaşırsak yaklaşalım açıktır ki sonuç hep bu limit için sıfır olacaktır. O halde eğer $f(0,0)=1$ olarak tanımlanırsa fonksiyon xy-düzleminin her noktasında sürekli olacaktır.

2) $\displaystyle\lim_{\substack{x\to 1\\ y\to\pi}} \frac{\cos(xy)}{1-x-\cos y} = -1$

$$1-\delta < x < 1+\delta$$
$$\pi-\delta < y < \pi+\delta$$

$\Rightarrow \quad |x-1|<\delta \;,\; |y-\pi|<\delta$ iken

$\left| \dfrac{\cos(xy)}{1-x-\cos y} - (-1) \right| < \varepsilon$ o.ş. $\delta=\delta(\varepsilon)>0$ var mıdır?

$$\left| \frac{\cos(xy)}{1-x-\cos y} + 1 \right| = \left| \frac{\cos(xy)+1-x-\cos y}{1-x-\cos y} \right|$$

$$< \left| \frac{1+1-x-(-1)}{1-x-1} \right| = \left| \frac{3-x}{-x} \right|$$

$$< \left| \frac{x-3}{x} \right|$$

$$< \frac{1+\delta-3}{1-\delta} = \frac{\delta-2}{1-\delta} = \varepsilon \Rightarrow \delta - 2 = \varepsilon - \varepsilon\delta$$

$$\delta(1+\varepsilon) = \varepsilon + 2$$

$$\delta = \frac{\varepsilon+2}{1+\varepsilon} > 0$$

olduğundan fonksiyon $(1,\pi)$ noktasında limite sahiptir.

3) $\int_0^1 \int_{2x^2}^{1+x} xy \, dy dx$ integralini verildiği şekilde ve integrasyon sırasını değiştirerek hesaplayınız.

$$\int_0^1 \int_{2x^2}^{1+x} xy \, dy dx = \int_0^1 x \left[\int_{2x^2}^{1+x} y \, dy \right] dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \Big|_{2x^2}^{1+x} \right] dx$$

$$= \int_0^1 x \left[\frac{(1+x)^2}{2} - \frac{4x^4}{2} \right] dx$$

$$= \frac{1}{2} \int_0^1 x \left[1 + 2x + x^2 - 4x^4 \right] dx$$

$$= \frac{1}{2} \int_0^1 \left(x + 2x^2 + x^3 - 4x^5 \right) dx$$

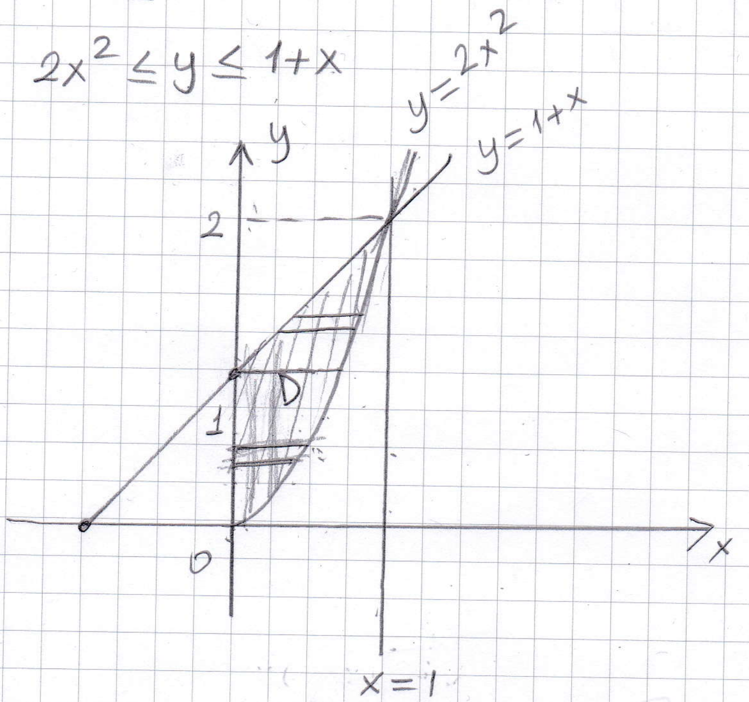
$$= \frac{1}{2} \left(\frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - 4\frac{x^6}{6} \Big|_0^1 \right)$$

$$= \frac{1}{2} \left(\frac{1}{2} + \cancel{\frac{2}{3}} + \frac{1}{4} - \cancel{\frac{4}{6}} \right)$$

$$= \frac{3}{8}$$

$0 \leq x \leq 1$

$2x^2 \leq y \leq 1+x$



$$I = \int_0^1 \int_0^{\sqrt{\frac{y}{2}}} xy \, dx\, dy + \int_1^2 \int_{y-1}^{\sqrt{\frac{y}{2}}} xy \, dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} \Big|_0^{\sqrt{\frac{y}{2}}} \right) y \, dy + \int_1^2 \left(\frac{x^2}{2} \Big|_{y-1}^{\sqrt{\frac{y}{2}}} \right) y \, dy$$

$$= \int_0^1 \left[\frac{y}{4} \right] \cdot y \, dy + \int_1^2 \left[\frac{y}{4} - \frac{(y-1)^2}{2} \right] y \, dy$$

$$= \int_0^1 \frac{y^2}{4} dy + \int_1^2 \frac{y^2}{4} dy - \int_1^2 \frac{(y^2-2y+1) \cdot y}{2} dy$$

$$= \frac{y^3}{12} \Big|_0^1 + \frac{y^3}{12} \Big|_1^2 - \frac{1}{2} \left(\frac{y^4}{4} - 2\frac{y^3}{3} + \frac{y^2}{2} \Big|_1^2 \right)$$

$$= \cancel{\frac{1}{12}} + \frac{8}{12} - \cancel{\frac{1}{12}} - \frac{1}{2} \left[\left(\frac{16}{4} - 2 \cdot \frac{8}{3} + \frac{4}{2} \right) - \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) \right]$$

$$= \frac{2}{3} - \frac{1}{2} \left[4 - \frac{16}{3} + 2 - \frac{1}{4} + \frac{2}{3} - \frac{1}{2} \right]$$

$$= \frac{2}{3} - 2 + \frac{8}{3} - 1 + \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \cancel{3} - \cancel{2} - \cancel{1} + \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

4) $\int_1^2 \int_0^{2y} (x+y^2)\,dx\,dy$ integralini verildiği şekilde ve integrasyon sırasını değiştirerek hesaplayınız.

$$\int_1^2 \left(\frac{x^2}{2}+y^2x \Big|_0^{2y}\right) dy = \int_1^2 \left[\frac{(2y)^2}{2}+y^2\cdot(2y)\right] dy$$

$$= \int_1^2 \left(\frac{4y^2}{2}+2y^3\right) dy$$

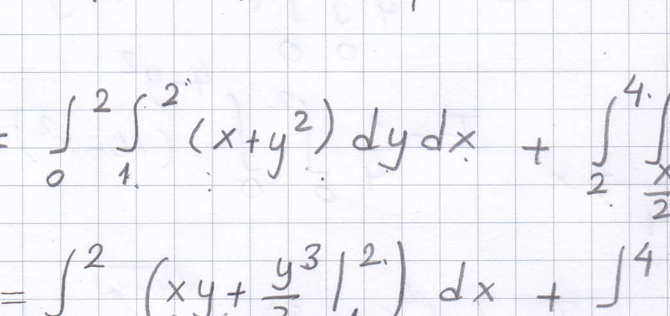
$$= 2\int_1^2 (y^2+y^3)\,dy$$

$$= 2\left[\frac{y^3}{3}+\frac{y^4}{4}\Big|_1^2\right]$$

$$= 2\left[\left(\frac{8}{3}+\frac{16}{4}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)\right]$$

$$= 2\left[\underset{(4)}{\frac{7}{3}}+\underset{(3)}{\frac{15}{4}}\right] = 2\cdot\frac{73}{12}$$

$$= \frac{73}{6}$$



$$I = \int_0^2 \int_1^2 (x+y^2)\,dy\,dx + \int_2^4 \int_{\frac{x}{2}}^2 (x+y^2)\,dy\,dx$$

$$= \int_0^2 \left(xy+\frac{y^3}{3}\Big|_1^2\right) dx + \int_2^4 \left(xy+\frac{y^3}{3}\Big|_{\frac{x}{2}}^2\right) dx$$

$$= \int_0^2 \left[\left(2x+\frac{8}{3}\right)-\left(x+\frac{1}{3}\right)\right] dx + \int_2^4 \left[\left(2x+\frac{8}{3}\right)-\left(\frac{x^2}{2}+\frac{x^3}{24}\right)\right] dx$$

$$I = \int_0^2 \left(x+\frac{7}{3}\right) dx + \int_2^4 \left(2x+\frac{8}{3}-\frac{x^2}{2}-\frac{x^3}{24}\right) dx$$

$$= \frac{x^2}{2}+\frac{7x}{3}\Big|_0^2 + \left(x^2+\frac{8x}{3}-\frac{x^3}{6}-\frac{x^4}{4\cdot 24}\Big|_2^4\right)$$

$$= \left(\frac{4}{2}+\frac{14}{3}\right) + \left(16+\frac{32}{3}-\frac{64}{6}-\frac{256}{4\cdot 24}\right) - \left(4+\frac{16}{3}-\frac{8}{6}-\frac{16}{4\cdot 24}\right)$$

$$= 2+\frac{14}{3}+16+\cancel{\frac{32}{3}}-\cancel{\frac{32}{3}}-\frac{64}{24}-4-\frac{16}{3}+\frac{4}{3}+\frac{4}{24}$$

$$= \cancel{2}+\frac{14}{3}+\cancel{16}-\frac{8}{3}-\cancel{4}-\frac{16}{3}+\frac{4}{3}+\frac{1}{6}$$

$$= \frac{14-8+4-16}{3}+\frac{1}{6}+14$$

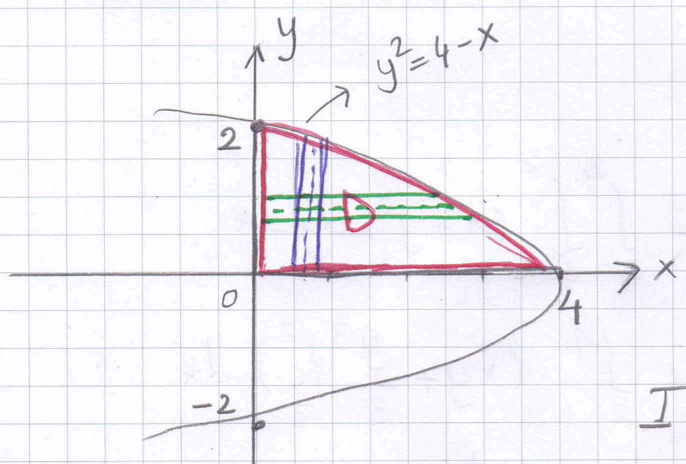
$$= -2+\frac{1}{6}+14$$

$$= 12+\frac{1}{6} = \frac{73}{6}$$

5) D bölgesi $y^2=4-x$, $x=0$, $y=0$ eğrilerinin sınırladığı bölge olduğuna göre bu bölge üzerinde

$$I = \iint_D \frac{1}{4}(16-x^2)\, dA$$

integralini hesaplayınız.



$$I = \frac{1}{4}\int_0^4\int_0^{\sqrt{4-x}} (16-x^2)\, dy\, dx \quad (x'e\ göre)$$

$$I = \frac{1}{4}\int_0^2\int_0^{4-y^2} (16-x^2)\, dx\, dy \quad (y'ye\ göre)$$

$$I = \frac{1}{4}\int_0^4 (16-x^2)\, y\Big|_0^{\sqrt{4-x}} dx$$

$$= \frac{1}{4}\int_0^4 (16-x^2)\sqrt{4-x}\, dx$$

$$= \frac{1}{4}\int_0^4 (4-x)(4+x)\sqrt{4-x}\, dx$$

$4 - x = u \Rightarrow x = 4 - u \Rightarrow x + 4 = 8 - u \quad \begin{cases} x = 4 \Rightarrow u = 0 \\ x = 0 \Rightarrow u = 4 \end{cases}$

$-dx = du \Rightarrow dx = -du$

$$\Rightarrow I = \frac{1}{4} \int_4^0 u \cdot (8-u) \sqrt{u} \, (-du)$$

$$= \frac{1}{4} \int_0^4 u^{3/2} (8-u) \, du$$

$$= \frac{1}{4} \int_0^4 8 u^{3/2} du - \frac{1}{4} \int_0^4 u^{5/2} du$$

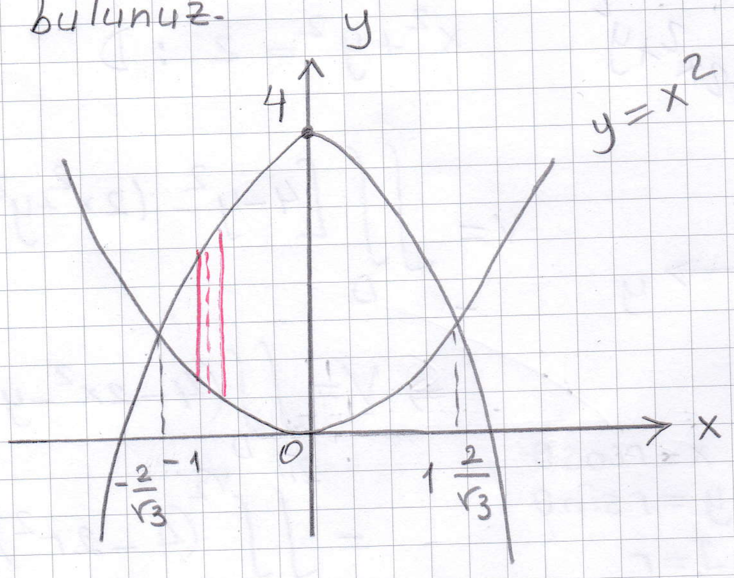
$$= \left(2 \cdot \frac{2}{5} u^{5/2} \Big|_0^4 \right) - \left(\frac{1}{4} \cdot \frac{2}{7} u^{7/2} \Big|_0^4 \right)$$

$$= \frac{4}{5} \left(4^{5/2} - 0 \right) - \frac{1}{14} \left(4^{7/2} - 0 \right)$$

$$= \frac{32 \cdot 4}{5} - \frac{1}{14} \cdot 128$$

$$= \frac{128}{5} - \frac{64}{7} = \frac{576}{35}$$

6) $y = x^2$ ve $y = 4 - 3x^2$ eğrilerinin sınırladığı alanı bulunuz.



$x^2 = 4 - 3x^2$

$\Rightarrow 4x^2 = 4$

$x = \mp 1$

$$A = \int_{-1}^{1} \int_{x^2}^{4-3x^2} dy\, dx$$

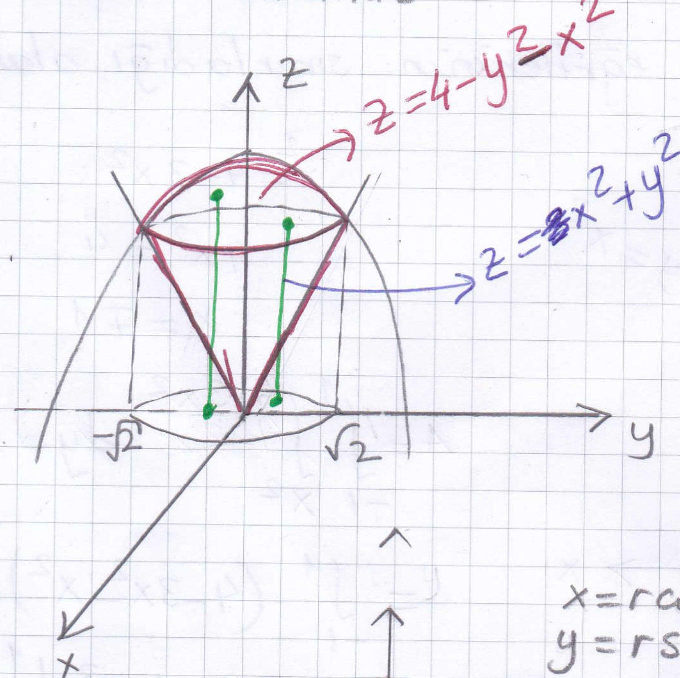
$$= \int_{-1}^{1} (4 - 3x^2 - x^2) dx$$

$$= 4x - 4\frac{x^3}{3} \Big|_{-1}^{1}$$

$$= \left(4 - \frac{4}{3}\right) - \left(-4 + \frac{4}{3}\right)$$

$$= \frac{16}{3} \, br^2$$

7) $z = x^2 + y^2$ ve $z = 4 - y^2 - x^2$ yüzeylerinin sınırladığı hacmi bulunuz.



$$x^2 + y^2 = 4 - y^2 - x^2$$

$$2x^2 + 2y^2 = 4$$

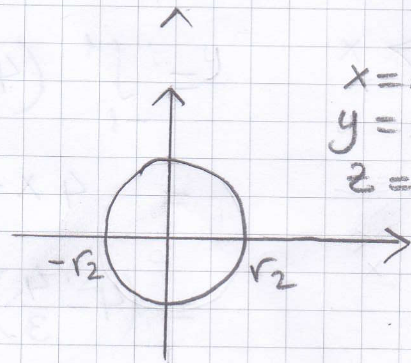
$$x^2 + y^2 = 2$$

$$V = \iint_D \left[4 - x^2 - y^2 - (x^2 + y^2)\right] dA$$

$$= \iint \left[4 - 2(x^2 + y^2)\right] dA$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = r$

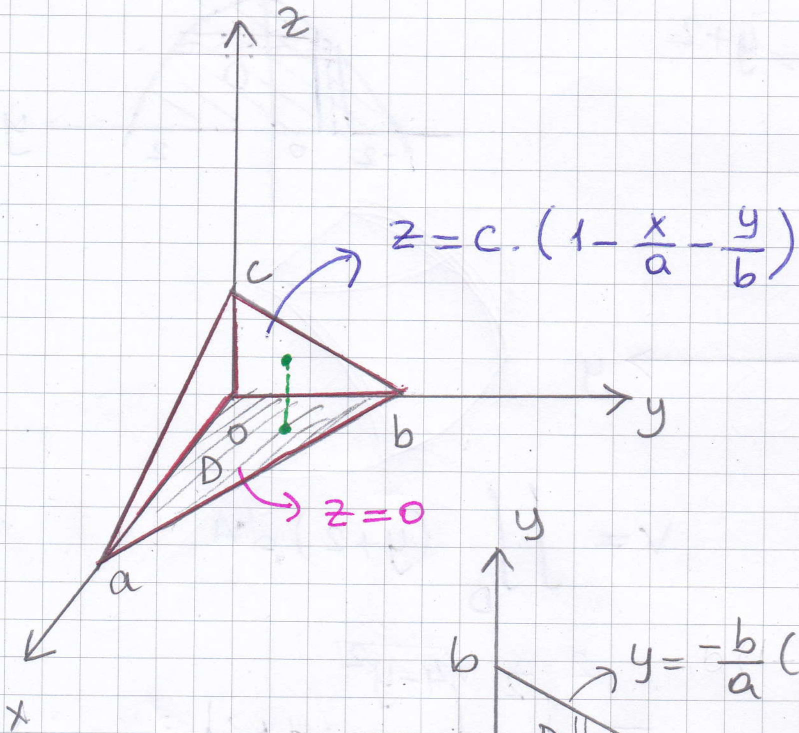
$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4 - 2r^2)\, r\, dr\, d\theta$$



$$= \int_0^{2\pi} \left(4\frac{r^2}{2} - 2\frac{r^4}{4}\Big|_0^{\sqrt{2}}\right) d\theta$$

$$= \int_0^{2\pi} (4 - 2)\, d\theta = 4\pi\, br^3$$

8) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ düzleminin koordinat düzlemleri ile sınırladığı hacmi hesaplayınız.

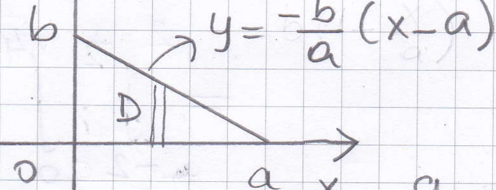


$(a,0) \to (0,b)$

$$\frac{y-0}{b-0}=\frac{x-a}{0-a}$$

$$-ay=b(x-a)$$

$$y=-\frac{b}{a}(x-a)$$



$$V=\iint_D c\left[1-\frac{x}{a}-\frac{y}{b}\right]dA = c\int_0^a\int_0^{\frac{b}{a}(a-x)}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a\left[y-\frac{x}{a}y-\frac{y^2}{2b}\Big|_0^{\frac{b}{a}(a-x)}\right]dx$$

$$=c\int_0^a\left[\frac{b}{a}(a-x)-\frac{x}{a}\cdot\frac{b}{a}(a-x)-\frac{1}{2b}\frac{b^2}{a^2}(a-x)^2\right]dx$$

$$=c\int_0^a\left[b-\frac{bx}{a}-\frac{xb}{a}+\frac{x^2b}{a^2}-\frac{b}{2a^2}(a^2-2ax+x^2)\right]dx$$

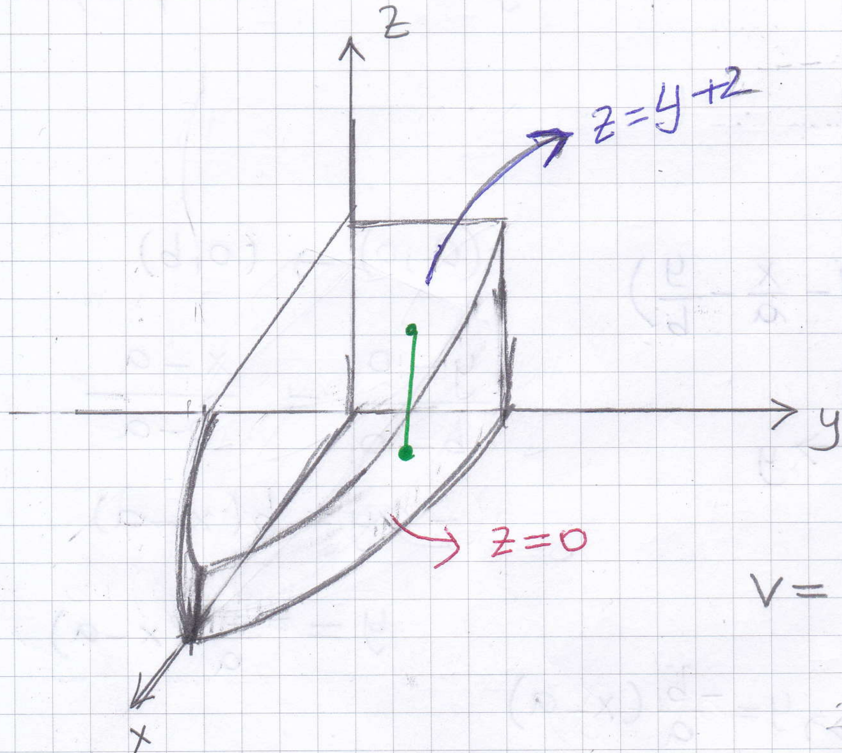
$$=c\int_0^a\left[b-\cancel{\frac{2bx}{a}}+\frac{x^2b}{a^2}-\frac{b}{2}+\cancel{\frac{2bx}{a}}-\frac{bx^2}{2a^2}\right]dx$$

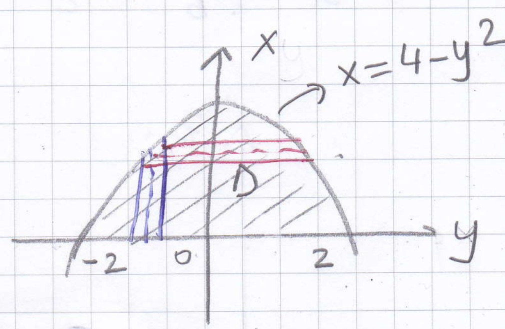
$$=c\int_0^a\left(\frac{b}{2}+\frac{bx^2}{2a^2}\right)dx$$

$$=c\left[\frac{bx}{2}+\frac{b}{2a^2}\cdot\frac{x^3}{3}\Big|_0^a\right]=c\left[\frac{ab}{2}+\frac{b}{2a^2}\cdot\frac{a^3}{3}\right]$$

$$=c\left[\frac{ab}{2}+\frac{ab}{6}\right]=\frac{4abc}{6}$$

9) $x=0$, $z=0$, $y^2=4-x$, $z=y+2$ yüzeylerinin sınırladığı hacmi hesaplayınız.





$$V=\iint_D (y+2)\,dA$$

$$=\int_{-2}^{2}\int_{0}^{4-y^2}(y+2)\,dx\,dy$$

$$=\int_{-2}^{2}(y+2)\left(x\Big|_0^{4-y^2}\right)dy$$

$$=\int_{-2}^{2}(y+2)(4-y^2)\,dy$$

$$=\int_{-2}^{2}(4y-y^3+8-2y^2)\,dy$$

$$=2y^2-\frac{y^4}{4}+8y-\frac{2}{3}y^3\Big|_{-2}^{2}$$

$$=\left(8-4+16-\frac{16}{3}\right)-\left(8-4-16+\frac{16}{3}\right)$$

$$=32-\frac{32}{3}$$

$$=\frac{64}{3}\ br^3$$