$$I = \int_1^2 \int_0^{\sqrt{3-y}} \frac{x}{y}\, dx\, dy$$ integralini

a) verildiği şekliyle hesaplayınız

b) İntegrasyon sırasını değiştirerek integrali yazınız.

a) $$\int_1^2 \int_0^{\sqrt{3-y}} \frac{x}{y}\, dx\, dy = \int_1^2 \left(\frac{x^2}{2y} \Big|_0^{\sqrt{3-y}} \right) dy = \int_1^2 \left(\frac{3-y}{2y} \right) dy = \frac{3}{2} \int_1^2 \frac{dy}{y} - \frac{1}{2} \int_1^2 dy$$

$$= \frac{3}{2} \ln|y| \Big|_1^2 - \left(\frac{y}{2} \Big|_1^2 \right)$$

$$= \frac{3}{2} \ln 2 - \left(1 - \frac{1}{2}\right)$$

$$= \frac{3}{2} \ln 2 - \frac{1}{2}$$

$1 = 3 - x^2 \Rightarrow x^2 = 2 \Rightarrow x = \mp\sqrt{2}$

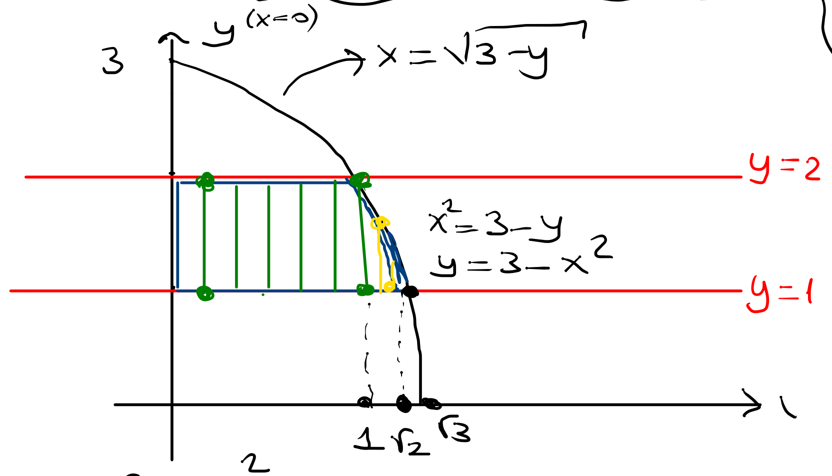
$1 \le y \le 2$

$0 \le x \le \sqrt{3-y}$

$y = 3 - x^2$

$2 = 3 - x^2$

$x^2 = 1 \Rightarrow x = \mp 1$



$$I = \int_0^1 \int_1^2 \frac{x}{y}\, dy\, dx + \int_1^{\sqrt{2}} \int_1^{3-x^2} \frac{x}{y}\, dy\, dx$$

$$I=\int_0^3\int_1^{e^y}(x+y)\,dx\,dy$$ integralini a) verildiği şekliyle hesaplayınız.

b) İntegrasyon sırasını değiştirerek integrali yazınız.

a) $$\int_0^3\int_1^{e^y}(x+y)\,dx\,dy=\int_0^3\left(\frac{x^2}{2}+yx\Big|_1^{e^y}\right)dy=\int_0^3\left(\frac{e^{2y}}{2}+ye^y-\frac{1}{2}-y\right)dy$$

$y=u \Rightarrow dy=du$

$e^y dy=dv \Rightarrow e^y=v$

$$=\frac{1}{2}\int_0^3 e^{2y}\,dy+\int_0^3 ye^y\,dy-\frac{1}{2}\int_0^3 dy-\int_0^3 y\,dy$$

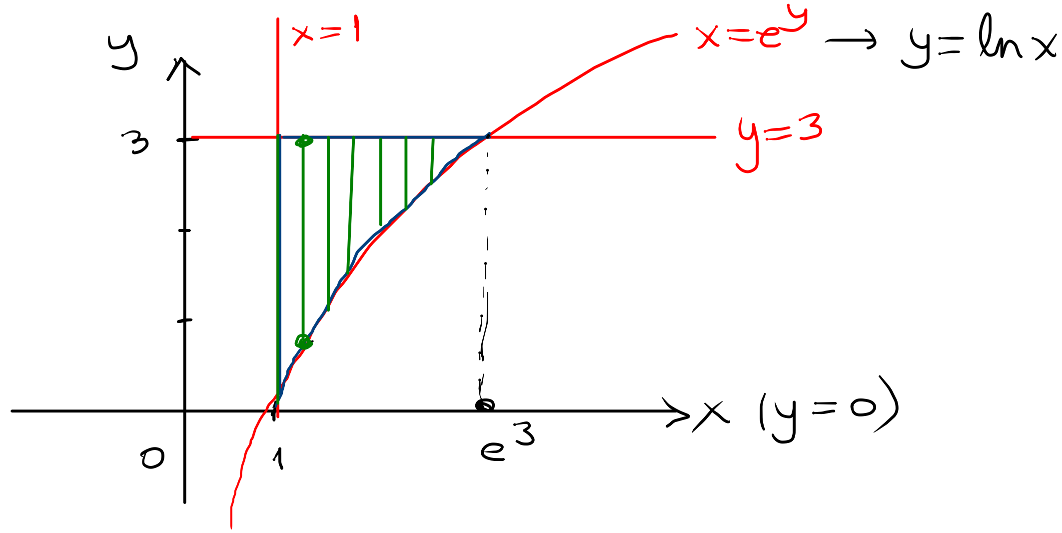
$$=\frac{1}{2}\cdot\frac{1}{2}e^{2y}\Big|_0^3+\left[ye^y\Big|_0^3-\int_0^3 e^y\,dy\right]-\left(\frac{y}{2}\Big|_0^3\right)-\left(\frac{y^2}{2}\Big|_0^3\right)$$

$$=\frac{1}{4}(e^6-1)+\left[3e^3-\left(e^y\Big|_0^3\right)\right]-\frac{3}{2}-\frac{9}{2}$$

$$=\frac{e^6}{4}-\frac{1}{4}+3e^3-e^3+1-6$$

$$=\frac{e^6}{4}+2e^3-\frac{21}{4}$$

$$I=\int_0^3\int_1^{e^y}(x+y)\,dx\,dy$$



$$0\le y\le 3$$

$$1\le x\le e^y$$

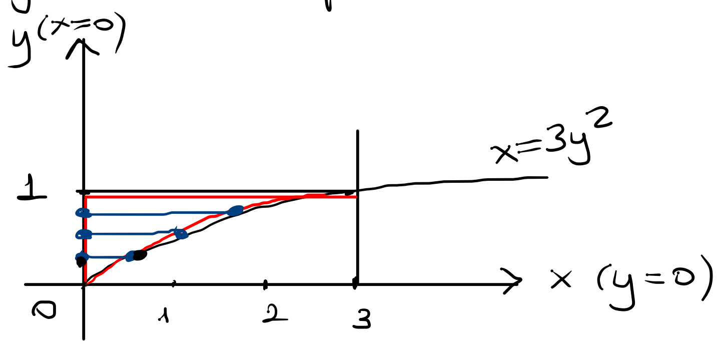
$$I=\int_1^{e^3}\int_{\ln x}^{3}(x+y)\,dy\,dx$$

$$I=\int_0^3\int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\,dy\,dx$$ integralini hesaplayınız.

$$0\le x\le 3$$

$$\sqrt{\frac{x}{3}}\le y\le 1$$

$$x=3y^2$$



$$y^3=t \Rightarrow 3y^2\,dy=dt$$

$$y=0\Rightarrow t=0$$

$$y=1\Rightarrow t=1$$

$$I=\int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$=\int_0^1\left(e^{y^3}x\Big|_0^{3y^2}\right)dy$$

$$=\int_0^1 3y^2e^{y^3}\,dy=\int_0^1 e^t\,dt$$

$$=e^t\Big|_0^1=e-1$$

İki katlı integrallerde değişken dönüşümü

1) Kartezyen koordinatlardan kartezyen koordinatlara değişken dönüşümü

$I = \iint_D f(x,y)\,\underbrace{dA}_{\substack{dx\,dy \\ \text{veya } dy\,dx}}$ integralinde $x = x(u,v)$, $y = y(u,v)$ değişken dönüşümü yapıldığında

D bölgesi bir D' bölgesine dönüşür ve integralin şekli

(Fonksiyonel Determinant) Jakobien

$$I = \iint_{D'} f\left[x(u,v),\, y(u,v)\right] |J| . \underbrace{dA'}_{\substack{du\,dv \\ \text{veya } dv\,du}}$$

olur. Burada

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{D(x,y)}{D(u,v)}$$

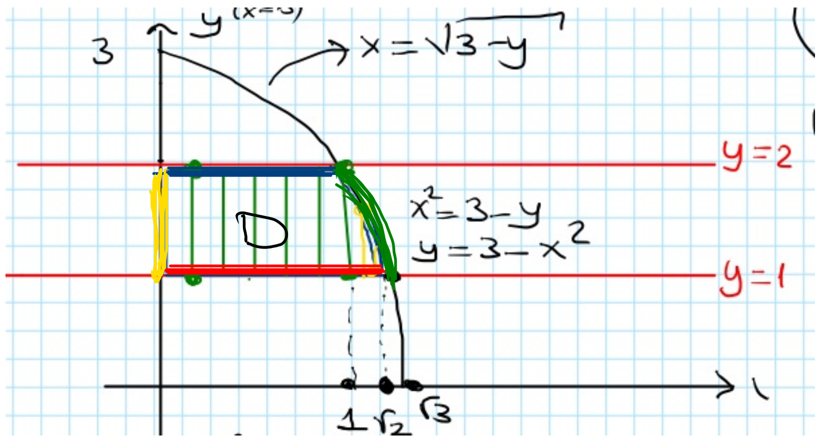
şeklindedir.

$$\frac{1}{J} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J . \frac{1}{J} = 1$$

Ör/ $I=\int_1^2\int_0^{\sqrt{3-y}} \frac{x}{y}\,dx\,dy$ integralinde $x^2=u-v$, $y=v$ değişken dönüşümünü yaparak integralini yazınız.

$$\underbrace{x^2=u-v \quad , \quad y=v}\ \rightarrow\ \begin{array}{l} x^2+y=u \\ y=v \end{array}$$



$1\le y\le 2$

$0\le x\le \sqrt{3-y}$

$\boxed{y=1}\Rightarrow\boxed{v=1}$

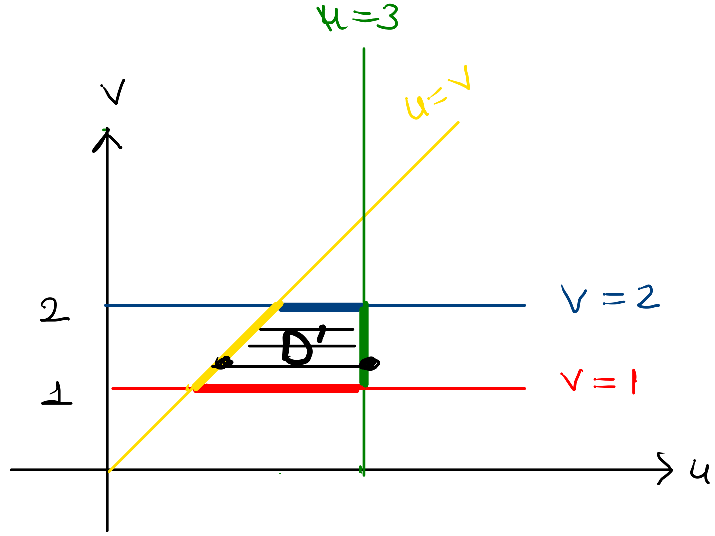
$\boxed{y=2}\Rightarrow\boxed{v=2}$

$\boxed{x=0}\Rightarrow u-v=0\Rightarrow\boxed{u=v}$

$x=\sqrt{3-y}\Rightarrow u-v=3-y$

$\Rightarrow u-v=3-v$

$\Rightarrow u=3$



I

$$\frac{1}{J}=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}2x & 1\\ 0 & 1\end{vmatrix}=2x$$

$$\Rightarrow J=\frac{1}{2x}$$

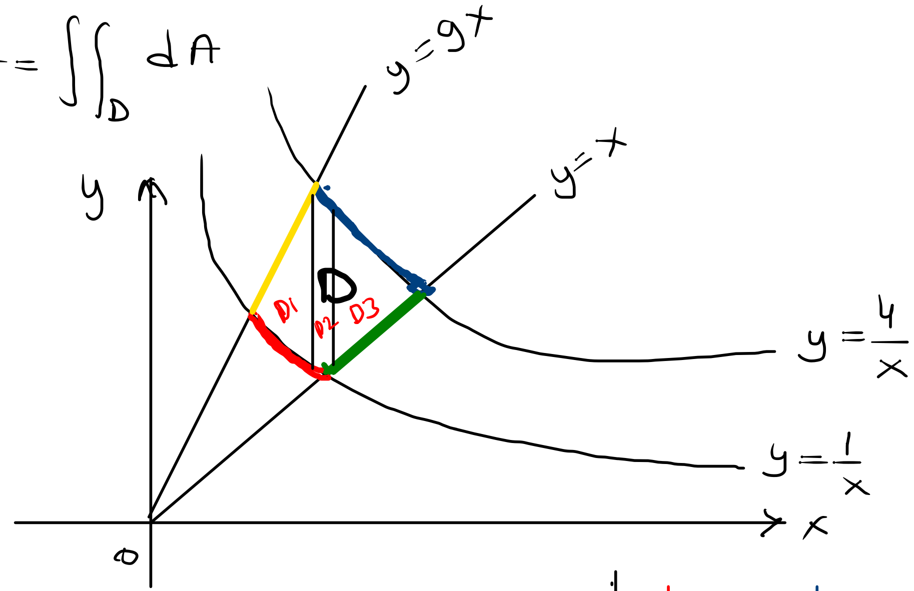
$$I=\iint_{D'}\frac{\cancel{x}}{y}\cdot\underbrace{\frac{1}{2\cancel{x}}}_{|J|}\,du\,dv=\frac{1}{2}\iint_{D'}\frac{1}{y}\,du\,dv=\frac{1}{2}\int_1^2\int_v^3\frac{1}{v}\,du\,dv$$

Ör: D: $x \cdot y = 1$, $xy = 4$, $\frac{y}{x} = 1$, $\frac{y}{x} = 9$ eğrileri ile sınırlı bölgenin 1. dörtte bir bölgede kalan kısmının alanını bulunuz.

$x \cdot y = 1 \Rightarrow y = \frac{1}{x}$ $\qquad$ $\frac{y}{x} = 1 \Rightarrow y = x$

$x \cdot y = 4 \Rightarrow y = \frac{4}{x}$ $\qquad$ $\frac{y}{x} = 9 \Rightarrow y = 9x$

$$A = \iint_D dA$$



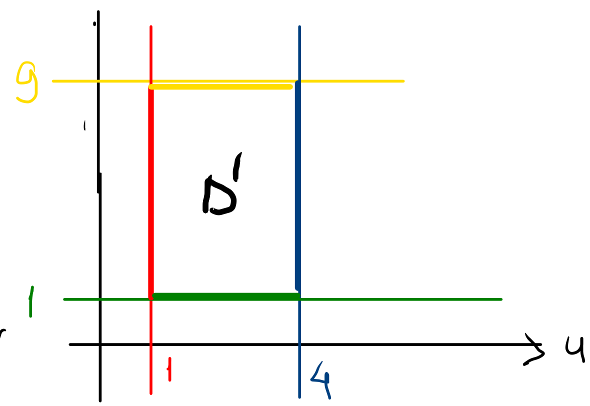
$x \cdot y = u \Rightarrow$ $\boxed{xy = 1 \Rightarrow u = 1}$

$\frac{y}{x} = v$ $\qquad$ $\boxed{xy = 4 \Rightarrow u = 4}$

$\Downarrow$

$\boxed{\frac{y}{x} = 1 \Rightarrow v = 1}$

$\boxed{\frac{y}{x} = 9 \Rightarrow v = 9}$



$$\frac{1}{J} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= \frac{y}{x} + \frac{y}{x}$$

$$= \frac{2y}{x}$$

$$A = \iint_{D'} |J| \, dA'$$

$$= \iint_{D'} \frac{x}{2y} \, dA' = \int_1^9 \int_1^4 \frac{1}{2v} \, du\, dv$$

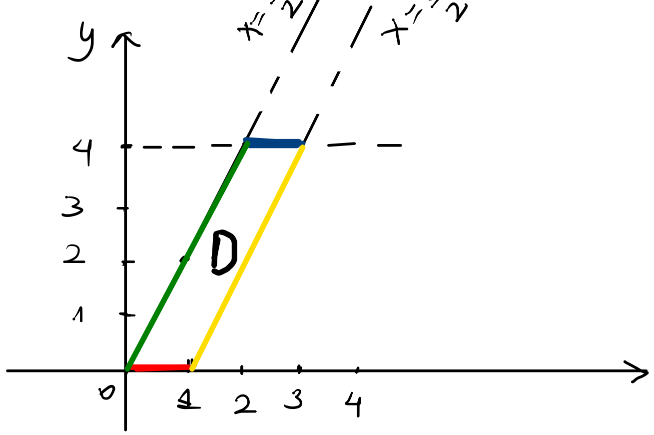
$$= \int_1^9 \int_1^4 \frac{1}{2v}\,du\,dv = \int_1^9 \left(\frac{u}{2v} \Big|_1^4 \right) dv = \frac{3}{2} \int_1^9 \frac{dv}{v} = \frac{3}{2} \ln|v| \Big|_1^9 = \frac{3}{2}(\ln 9 - \underbrace{\ln 1}_{0})$$

$$= \frac{3}{2} \ln 3^2 = 3 \ln 3$$

Ör/ $\displaystyle\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{2} \right) dx\,dy$ integralinde $y = 2v$, $x = \frac{u+v}{2}$ değişken dönüşümünü yaparak integrali yazınız.

$0 \le y \le 4$

$\frac{y}{2} \le x \le \frac{y}{2} + 1$



$\underline{y = 2v}$

- $y = 0 \Rightarrow 2v = 0 \Rightarrow v = 0$
- $y = 4 \Rightarrow 2v = 4 \Rightarrow v = 2$

$\underline{x = \frac{u+v}{2}}$

- $x = \frac{y}{2} \Rightarrow x = v$

$\Rightarrow v = \frac{u+v}{2}$

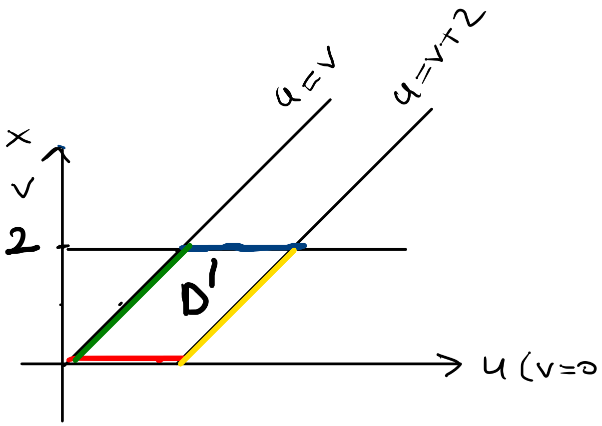
$\Rightarrow \frac{v}{2} = \frac{u}{2} \Rightarrow u = v$

- $x = \frac{y}{2} + 1 \Rightarrow x = v + 1$

$\Rightarrow v + 1 = \frac{u+v}{2}$

$\Rightarrow 2v + 2 = u + v$

$\Rightarrow v = u - 2 \Rightarrow u = v + 2$



$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 2 \end{vmatrix} = 1$$

$$I = \iint_{D'} \frac{u-v}{2} \cdot 1 \, du\,dv = \int_0^2 \int_v^{v+2} \frac{u-v}{2} \, du\,dv = \int_0^2 \left[\frac{u^2}{4} - \frac{uv}{2} \Big|_v^{v+2} \right] dv$$

$$= \int_0^2 \left[\frac{(v+2)^2}{4} - \frac{v(v+2)}{2} - \frac{v^2}{4} + \frac{v^2}{2} \right] dv$$

$$= \int_0^2 \left[\frac{\cancel{v^2} + 4v + 4 - \cancel{v^2}}{4} + \frac{\cancel{v^2} - \cancel{v^2} - 2v}{2} \right] dv$$

$$= \int_0^2 \left[(v+1) + (-v) \right] dv$$

$$= \int_0^2 dv = v \Big|_0^2 = 2$$

$$\frac{2x-y}{2} = x - \frac{y}{2} = \frac{u+v}{2} - v = \frac{u-v}{2}$$