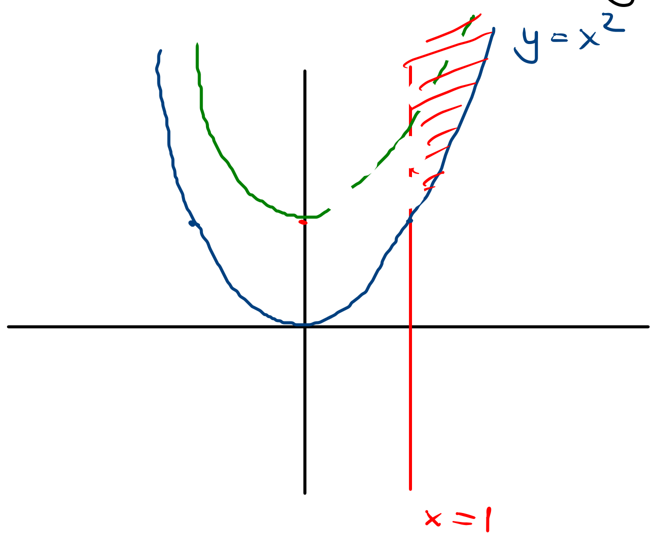
– Uygulama –

1) $z=(y-x^2)^{\ln(x-1)}$ tanım bölgesini bulup koordinat düzleminde gösteriniz.

$a>0 \quad a\neq 1 \qquad x-1>0 \rightarrow x>1$

$y-x^2>0 \quad y-x^2\neq 1$

$y>x^2 \qquad y\neq 1+x^2$



$D(z)=\{(x,y)\in\mathbb{R}^2 \mid y>x^2,\ y\neq 1+x^2,\ x>1\}$

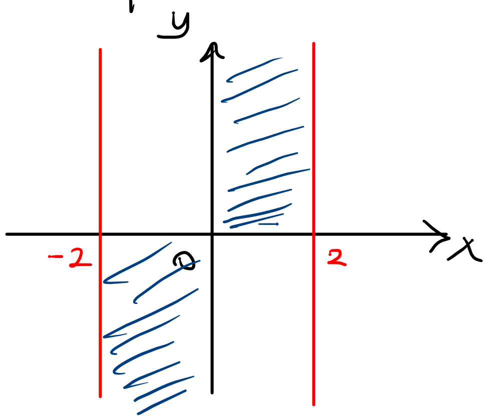
2) $f(x,y)=\arcsin\frac{x}{2}+\sqrt{xy}$ tanım bölgesini bulup koordinat düzleminde gösteriniz.

$-1\leq\frac{x}{2}\leq 1$

$-2\leq x\leq 2$

$xy\geq 0$

$x\geq 0 \quad x\leq 0$
$y\geq 0 \quad y\leq 0$



$D(z)=\{(x,y)\in\mathbb{R}^2 \mid -2\leq x\leq 2,\ xy\geq 0\}$

3) $f(x,y)=\dfrac{1}{\sqrt{9x^2-y^2}}$ tanım bölgesini bulup koordinat düzleminde gösteriniz.

$$9x^2-y^2>0$$

$$(3x-y)(3x+y)>0$$

$3x-y>0$

$3x+y>0$

1) $y<3x$

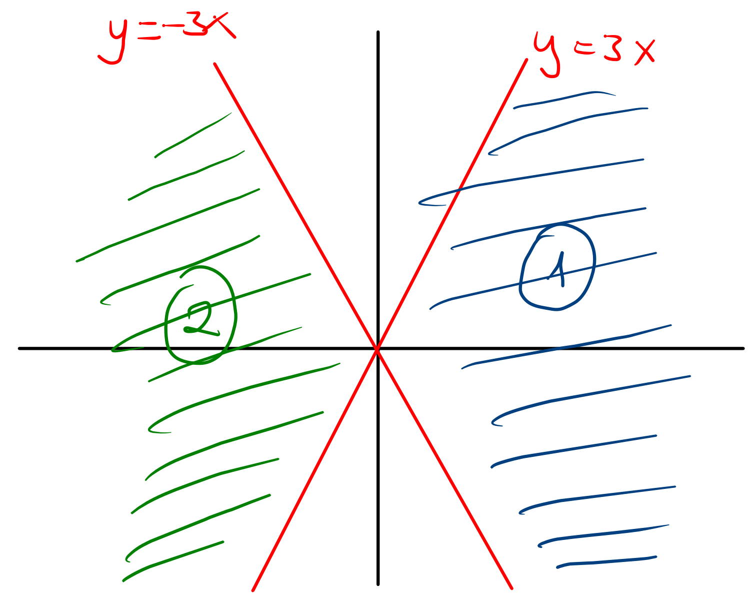
$y>-3x$

$3x-y<0$

$3x+y<0$

2) $y>3x$

$y<-3x$



$$D(z)=\{(x,y)\in\mathbb{R}^2 \mid 9x^2-y^2>0\}$$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y+xy)}{xy}\cdot \sin x\cdot\sin y=0$ olduğunu gösteriniz.

$\varepsilon>0$ verildiğinde $|x-0|<\delta$, $|y-0|<\delta$ iken $\left|\dfrac{x+y+xy}{xy}\sin x \sin y-0\right|<\varepsilon$ o.ş. $\delta=\delta(\varepsilon)>0$

$(\sqrt{x^2+y^2}<\delta)$

bulmalıyız.

$$\left|\frac{x+y+xy}{xy}\sin x \sin y\right| = \frac{|x+y+xy|}{|xy|}\cdot|\sin x||\sin y|$$

$|\sin x| \leq |x|$

$|\sin y| \leq |y|$

$$\leq \frac{|x+y+xy|}{\cancel{|x|}\cancel{|y|}}\,\cancel{|x|}\cdot\cancel{|y|}$$

$$\leq |x|+|y|+|x|.|y|$$

$$< \delta+\delta+\delta^2 = 2\delta+\delta^2 = \varepsilon$$

$$\delta^2+2\delta-\varepsilon = 0 \Rightarrow \delta_{1,2} = \frac{-2\mp\sqrt{4+4\varepsilon}}{2} = -1\mp\sqrt{1+\varepsilon}$$

$$\delta = -1+\sqrt{1+\varepsilon} > 0$$

NOT :

$[0,x]$

$f(x) = \sin x$

O.D.T - $f'(c) = \dfrac{\sin x - \cancel{\sin 0}^{\,0}}{x-0} = \cos c \Rightarrow \left|\dfrac{\sin x}{x}\right| = |\cos c| \leq 1$

$|\sin x| \leq |x|$

5) $(x,y)\neq(0,0)$ iken $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ dan $f(x,y)$ fonksiyonu $(0,0)$'da sürekli olacak şekilde tanımlanabilir mi?

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to kx}} \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}.$$

limit yoktur. Çünkü k'nın her farklı değeri için sonuç farklı olacaktır. Dolayısıyla fonksiyon orjinde süreksizdir. $f(0,0)$ tanımlanamaz.

6) $f(x,y)=\dfrac{x^2-y}{x+y}$ fonksiyonu $(2,0)$ noktasında sürekli midir?

$$\lim_{(x,y)\to(2,0)} f(x,y)=f(2,0) \quad \text{olmalı.}$$

$$f(2,0)=\frac{4-0}{2+0}=2 \checkmark$$

$\varepsilon>0$ verildiğinde $|x-2|<\delta$, $|y|<\delta$ iken $\left|\dfrac{x^2-y}{x+y}-2\right|<\varepsilon$ o.ş. bir $\delta=\delta(\varepsilon)>0$ bulmalıyız.

$$\left|\frac{x^2-y}{x+y}-2\right| = \left|\frac{x^2-y-2x-2y}{x+y}\right| = \left|\frac{x^2-2x-3y}{x+y}\right|$$

$|x-2|<\delta \Rightarrow -\delta < x-2 < \delta$, $|y|<\delta \Rightarrow -\delta<y<\delta$

$2-\delta < x < 2+\delta$

$(2-\delta)^2 < x^2 < (2+\delta)^2$

$4-2\delta < 2x < 4+2\delta$

$3\delta > -3y > -3\delta$

$-3\delta < -3y < 3\delta$

$$\left|\frac{x^2-2x-3y}{x+y}\right| < \frac{(2+\delta)^2-(4-2\delta)+3\delta}{2-\delta-\delta} = \frac{\cancel{4}+4\delta+\delta^2-\cancel{4}+2\delta+3\delta}{2-2\delta} = \frac{9\delta+\delta^2}{2-2\delta} < \frac{\overset{5}{\cancel{10}}\delta}{\cancel{2}(1-\delta)}$$

$0<\delta<1 \Rightarrow -\delta<\delta^2<\delta$

$\Rightarrow \frac{5\delta}{1-\delta}=\varepsilon$

$5\delta = \varepsilon-\varepsilon\delta$

$(5+\varepsilon)\delta=\varepsilon$

$\delta=\frac{\varepsilon}{5+\varepsilon}>0$

O halde $\lim_{(x,y)\to(2,0)} \frac{x^2-y}{x+y} = 2$ 'dir.

7) $\lim\limits_{(x,y)\to(0,0)} x\cdot\sin\frac{1}{y}$ limiti mevcut mudur?

$\forall t$ için $-1 \leq |\sin t| \leq 1$

$y \neq 0$

$$-1 \leq \sin\left(\frac{1}{y}\right) \leq 1$$

$$-x \leq x\sin\left(\frac{1}{y}\right) \leq x \quad \Rightarrow \lim_{x\to 0} x = \lim_{x\to 0} -x = 0 \Rightarrow \lim_{x\to 0} x\cdot\sin\left(\frac{1}{y}\right) = 0$$

$\varepsilon > 0$ verildiğinde $|x| < \delta$, $|y| < \delta$ iken $\left|x\cdot\sin\frac{1}{y} - 0\right| < \varepsilon$ o.ş. $\delta = \delta(\varepsilon) > 0$ bulmalıyız.

$$\left|x\cdot\sin\frac{1}{y}\right| = |x|\cdot\left|\sin\frac{1}{y}\right| \leq |x| < \delta = \varepsilon > 0$$

8) $f(x,y) = \dfrac{xy^2}{(x^2+y^2)^{3/2}}$ fonksiyonu $(0,0)$ da sürekli midir?

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \\ \left.\begin{array}{l} x\to 0 \\ y\to 0 \end{array}\right\} r\to 0 \end{array}\right\} \lim_{r\to 0} \frac{r\cos\theta\cdot r^2\sin^2\theta}{[r^2]^{3/2}} = \lim_{r\to 0} \frac{\cancel{r^3}\sin^2\theta\cdot\cos\theta}{\cancel{r^3}} = \sin^2\theta\cdot\cos\theta$$

sonuç θ'ya bağlı olduğundan limit mevcut değildir.

Dolayısıyla $f(x,y)$ $(0,0)$'da süreksizdir.

a) $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2-3xy+y^2}{2x^2+xy-y^2} = \lim\limits_{\substack{x\to 1\\ y=x+1}} \dfrac{2x^2-3x(x+1)+(x+1)^2}{2x^2+x(x+1)-(x+1)^2} = \lim\limits_{\substack{x\to 1\\ y=x+1}} \dfrac{\cancel{2x^2}-\cancel{3x^2}-3x+\cancel{x^2}+2x+1}{2x^2+x^2+x-x^2-2x-1}$

$= \lim\limits_{x\to 1} \dfrac{-x+1}{2x^2-x-1} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 1} \dfrac{-1}{4x-1} = \dfrac{-1}{3}$

$\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2-3xy+y^2}{2x^2+xy-y^2} = \lim\limits_{r\to 0} \dfrac{2(1+r\cos\theta)^2-3(1+r\cos\theta)(2+r\sin\theta)+(2+r\sin\theta)^2}{2(1+r\cos\theta)^2+(1+r\cos\theta)(2+r\sin\theta)-(2+r\sin\theta)^2}$

$x=1+r\cos\theta$

$y=2+r\sin\theta$

$= \lim\limits_{r\to 0} \dfrac{\cancel{r}\left[\sin\theta-2\cos\theta+2r\cos^2\theta+r\sin^2\theta-3r\cos\theta\sin\theta\right]}{\cancel{r}(6\cos\theta-3\sin\theta+2r\cos^2\theta-r\sin^2\theta+r\cos\theta\sin\theta)}$

$r\to 0 \Rightarrow x\to 1,\ y\to 2$

$= \dfrac{\sin\theta-2\cos\theta}{6\cos\theta-3\sin\theta} = \dfrac{\cancel{\sin\theta-2\cos\theta}}{-3(\cancel{\sin\theta-2\cos\theta})} = -\dfrac{1}{3}$

$\varepsilon>0$ verildiğinde $|x-1|<\delta,\ |y-2|<\delta$ iken $\left|\dfrac{2x^2-3xy+y^2}{2x^2+xy-y^2}+\dfrac{1}{3}\right|<\varepsilon$ o.ş.

$\delta=\delta(\varepsilon)$ bulmalıyız. $\left(\delta=\dfrac{3\varepsilon}{2+2\varepsilon}\right)$

10) $f(x,y)=\begin{cases} \dfrac{x^3y-xy^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

fonksiyonu için $(0,0)$'da Schwarz Teoreminin sağlanıp sağlanmadığını gösteriniz.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3\cdot 0-h\cdot 0}{h^2+0}-0}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0}\frac{\frac{0\cdot k-0\cdot k^3}{0+k^2}-0}{k} = 0$$

$$\frac{\partial f}{\partial x} = \begin{cases} \dfrac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-xy^3)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-xy^3)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{\frac{-k^5}{k^4}}{k} = -1$$

$\neq$ Schwarz teoremi sağlanmaz.

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5}{h^4}}{h} = 1$$

11) $z = f(x+y, x-y)$ olduğuna göre $\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2$ olduğunu gösteriniz.

$z = f(u,v)$

$u = x+y \Rightarrow u = u(x,y)$

$v = x-y \Rightarrow v = v(x,y)$

$\Big\}\; z = f(u,v) = f(u(x,y), v(x,y)) = F(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot(-1)$$

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = \left[\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]\left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right] = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2$$

12) $u = f(x-y,\ y-z,\ z-x)$ olduğuna göre $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ olduğunu gösteriniz.

$u = f(r, s, t)$

$$\left.\begin{array}{ll} r = x-y & r = r(x,y) \\ s = y-z & s = s(y,z) \\ t = z-x & t = t(x,z) \end{array}\right\} \quad u = f\left[r(x,y), s(y,z), t(x,z)\right] = F(x,y,z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}\cdot 1 + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\cdot(-1) = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}\cdot 1 + \frac{\partial u}{\partial t}\cdot 0 = -\cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} = \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}\cdot 1 = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$$

$+$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

13) $f(x,y) = \dfrac{x^2+y^2-x^3y^3}{x^2+y^2}$ $(x,y) \neq (0,0)$ fonksiyonu orjinde nasıl tanımlanmalıdır ki tüm reel xy-düzleminde sürekli olsun.

14) $\displaystyle\lim_{\substack{x\to 1\\ y\to \pi}} \frac{\cos(xy)}{1-x-\cos y} = -1$ olduğunu gösteriniz.