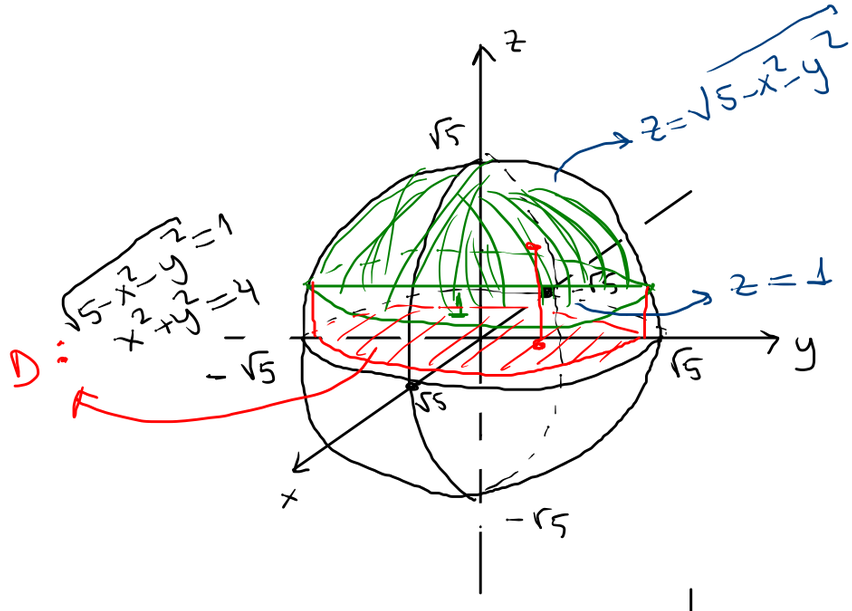
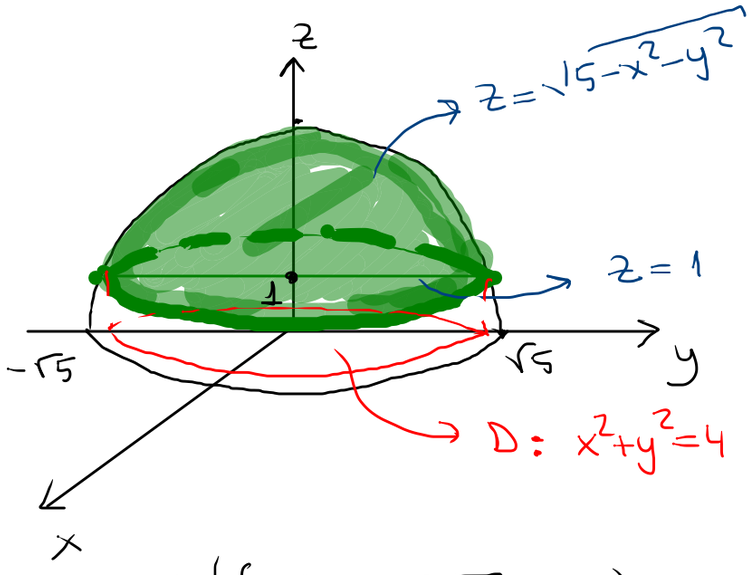
Örnekler

1) $x^2+y^2+z^2=5$ küresinin $z=1$ düzlemi üstünde kalan kısmının hacmini bulunuz.





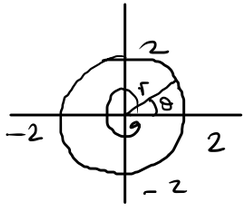
$\sqrt{5-x^2-y^2}=1$

$\Rightarrow x^2+y^2=4$

$x=r\cos\theta$

$y=r\sin\theta$

$J=r$



$$V=\iint_D \left(\sqrt{5-x^2-y^2}-1\right)dA$$

$$=\int_0^{2\pi}\int_0^2 \left(\sqrt{5-r^2}-1\right) r\,dr\,d\theta$$

$$V=\iint_D \left(\sqrt{5-x^2-y^2}-1\right) dA$$

$$=\int_0^{2\pi}\int_0^2 \left(\sqrt{5-r^2}-1\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{5-r^2}\, dr\, d\theta - \int_0^{2\pi}\int_0^2 r\, dr\, d\theta$$

$5-r^2=t$

$-2r\,dr = dt$

$r\,dr = \frac{-dt}{2}$

$r=0 \Rightarrow t=5$

$r=2 \Rightarrow t=1$

$$= \int_0^{2\pi}\int_5^1 \sqrt{t}\cdot\left(\frac{-dt}{2}\right) d\theta - \int_0^{2\pi}\left[\frac{r^2}{2}\Big|_0^2\right] d\theta$$

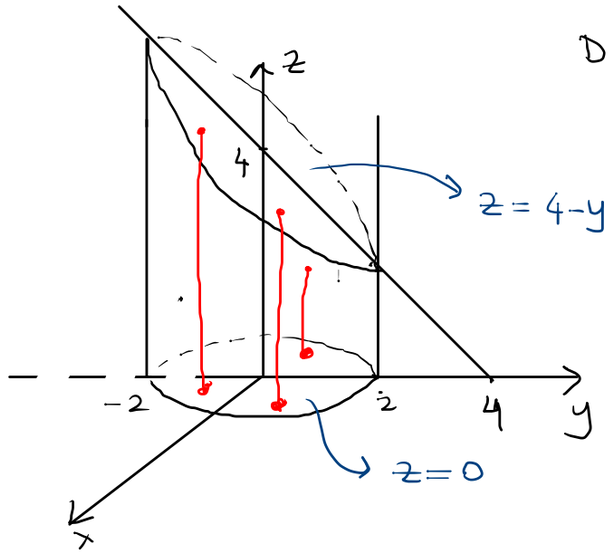
$$= \frac{1}{2}\int_0^{2\pi}\int_1^5 \sqrt{t}\, dt\, d\theta - \int_0^{2\pi}(2-0)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{2}{3}t^{3/2}\Big|_1^5\right) d\theta - \left(2\theta\Big|_0^{2\pi}\right)$$

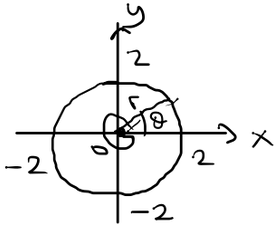
$$= \frac{1}{3}(5\sqrt{5}-1)\,\theta\Big|_0^{2\pi} - 4\pi$$

$$= \frac{2\pi}{3}(5\sqrt{5}-1) - 4\pi = \frac{10\sqrt{5}\pi - 14\pi}{3}\ br^3$$

2) $x^2+y^2=4$ silindir yüzeyi ile $y+z=4$ ve $z=0$ düzlemleri arasında kalan cismin hacmini bulunuz.



$D: x^2+y^2=4$



$x = r\cos\theta$

$y = r\sin\theta$

$J = r$

$$V = \iint_D (4-y-0)\,dA = \int_0^{2\pi}\int_0^2 (4-r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r\,dr\,d\theta - \int_0^{2\pi}\int_0^2 r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(2r^2\Big|_0^2\right) d\theta - \int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta\Big|_0^2\right] d\theta$$

$$= 8\int_0^{2\pi} d\theta - \frac{8}{3}\int_0^{2\pi} \sin\theta\,d\theta$$

$$= 8\theta\Big|_0^{2\pi} + \frac{8}{3}\cos\theta\Big|_0^{2\pi}$$

$$= 8(2\pi-0) + \frac{8}{3}(1-1)$$

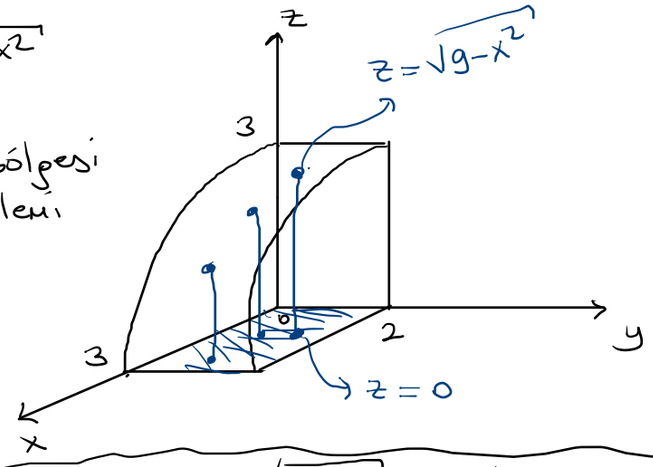
$$= 16\pi \text{ br}^3$$

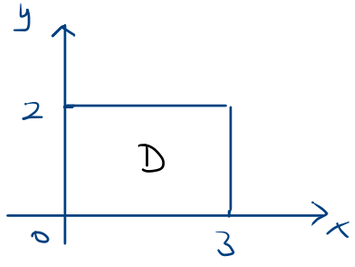
3) $x^2+z^2=9$ silindiri ile $y=2, y=0, x=0, z=0$ düzlemleri tarafından sınırlandırılan cismin hacmini bulunuz.

$z=\sqrt{9-x^2}$

↓ fonk.

izdüşüm bölgesi xoy düzlemi





$x=3\sin\theta$

$dx=3\cos\theta\,d\theta$

$x=0 \Rightarrow \sin\theta=0$

$\theta=0$

$x=3 \Rightarrow 3\sin\theta=3$

$\sin\theta=1$

$\theta=\pi/2$

$$V=\iint_D \left(\sqrt{9-x^2}-0\right)dA$$

$$=\int_0^3\int_0^2 \sqrt{9-x^2}\,dy\,dx$$

$$=\int_0^3 \left(\sqrt{9-x^2}\;y\,\Big|_0^2\right)dx$$

$$=2\int_0^3 \sqrt{9-x^2}\,dx$$

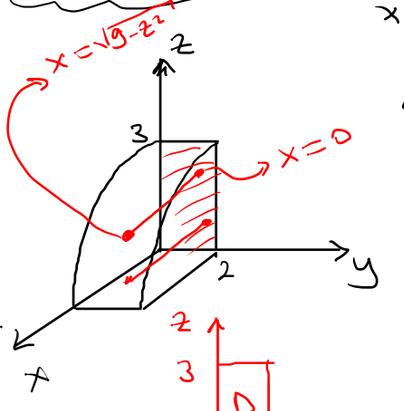
$$=2\int_0^{\pi/2} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

$$=18\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=18\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta=9\int_0^{\pi/2}d\theta+9\int_0^{\pi/2}\cos 2\theta\,d\theta$$

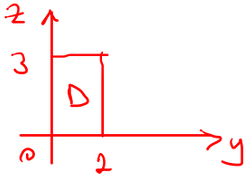
$$=9\theta\,\Big|_0^{\pi/2}+\frac{9}{2}\sin 2\theta\,\Big|_0^{\pi/2}$$

$$=\frac{9\pi}{2}\text{ br}^3$$



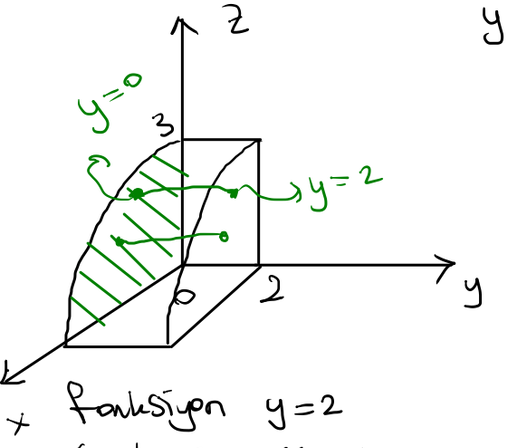
$x=\sqrt{9-z^2}$ → fonk.

izdüşüm bölgesi yoz olsun.

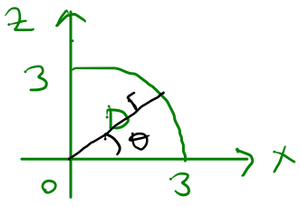


$$V=\iint_D \left(\sqrt{9-z^2}-0\right)dA$$

$$=\int_0^3\int_0^2 \sqrt{9-z^2}\,dy\,dz$$



$y=2$,



$x = r\cos\theta$

$y = r\sin\theta$

$\overline{J} = r$

fonksiyon $y=2$

izdüşüm bölgesi

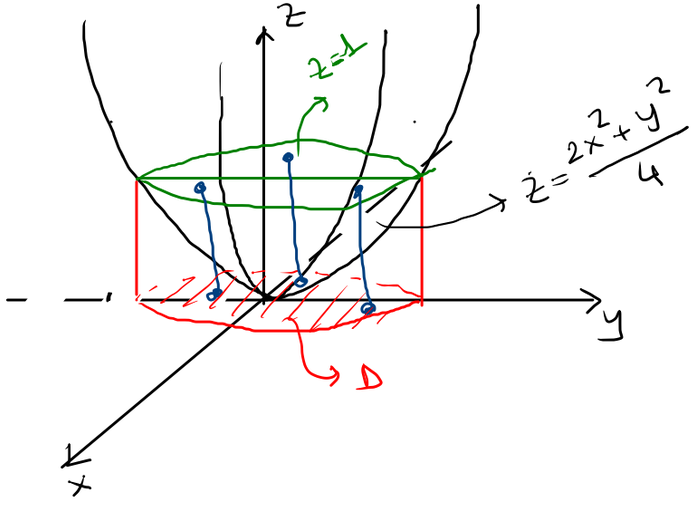
xOz

$$V = \iint_D (2-0)\,dA = 2\int_0^{\pi/2}\int_0^3 r\,dr\,d\theta = 2\int_0^{\pi/2}\left(\frac{r^2}{2}\Big|_0^3\right)d\theta$$

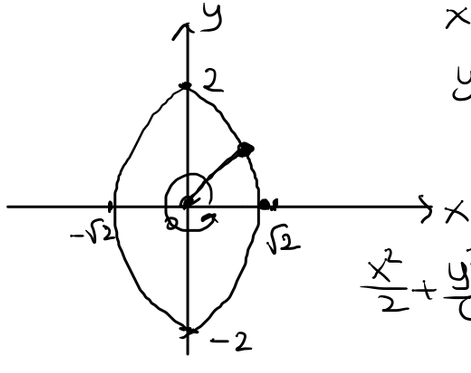
$$= 9\int_0^{\pi/2} d\theta$$

$$= 9\theta\Big|_0^{\pi/2} = \frac{9\pi}{2}\,br^3$$

4) $2x^2+y^2=4z$ paraboloidi ile $z=1$ düzlemi arasında kalan cismin hacmini bulunuz.



$$\frac{2x^2+y^2}{4} = 1 \Rightarrow \frac{x^2}{2}+\frac{y^2}{4}=1 \quad \text{Elips.}$$



$x = \sqrt{2}r\cos\theta$

$y = 2r\sin\theta$

$J = 2\sqrt{2}r$

$$V = \iint_D \left(1 - \frac{2x^2+y^2}{4}\right)dA =$$

$$\frac{x^2}{2}+\frac{y^2}{4}=1 \Rightarrow \frac{2r^2\cos^2\theta}{2}+\frac{4r^2\sin^2\theta}{4}=1$$

$$r^2 = 1 \Rightarrow r = 1$$

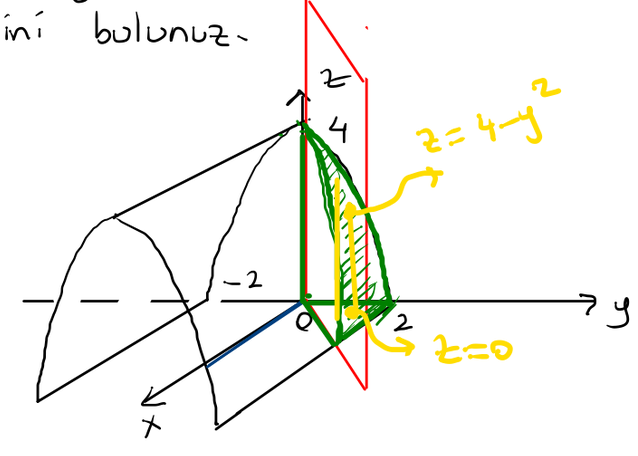
$$V = \iint_D \left(1 - \frac{2x^2+y^2}{4}\right) dA = \int_0^{2\pi}\int_0^1 \left[1 - \frac{2\cdot(2r^2\cos^2\theta) + 4r^2\sin^2\theta}{4}\right] \cdot 2\sqrt{2}\, r\, dr\, d\theta$$

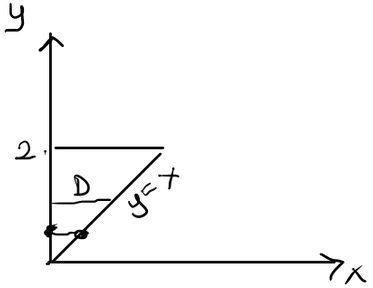
$$= 2\sqrt{2}\int_0^{2\pi}\int_0^1 \left[1 - r^2\right] \cdot r\, dr\, d\theta = 2\sqrt{2}\int_0^{2\pi}\int_0^1 \left[r - r^3\right] dr\, d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} \left[\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1\right] d\theta = 2\sqrt{2}\int_0^{2\pi} \left[\frac{1}{2} - \frac{1}{4}\right] d\theta = \frac{2\sqrt{2}}{4}\int_0^{2\pi} d\theta$$

$$= \frac{\sqrt{2}}{2}\cdot\theta\Big|_0^{2\pi} = \frac{\sqrt{2}}{2}\cdot 2\pi = \sqrt{2}\,\pi \text{ br}^3$$

5) $x \geq 0$, $y = x$, $z = 0$ düzlemleri ile $z = 4 - y^2$ parabolik silindirinin birinci bölgede sınırladığı cismin hacmini bulunuz.



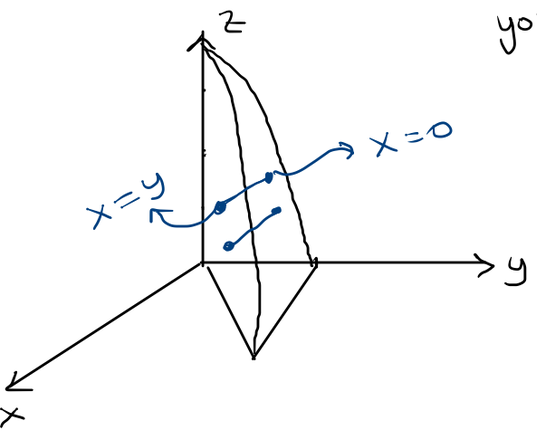


$$V = \iint_D (4 - y^2 - 0)\, dA$$

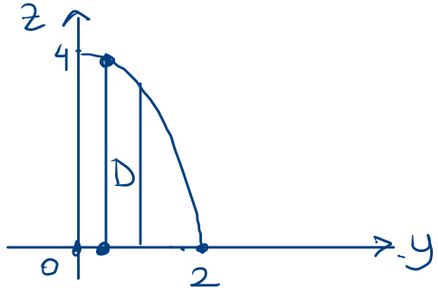
$$= \int_0^2 \int_0^y (4 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[(4 - y^2)x\Big|_0^y\right] dy = \int_0^2 (4y - y^3)\, dy$$

$$= 2y^2 - \frac{y^4}{4}\Big|_0^2 = 8 - 4 = 4 \text{ br}^3$$



yoz izdüşüm bölgesi seçilirse;



$$V = \iint_D (y-0)\, dA$$

$$= \int_0^2 \int_0^{4-y^2} y \, dz\, dy = \int_0^2 \left(yz \Big|_0^{4-y^2} \right) dy$$

$$= \int_0^2 \left[y(4-y^2-0) \right] dy$$

$$= \int_0^2 (4y - y^3)\, dy = 2y^2 - \frac{y^4}{4} \Big|_0^2 = 4 \text{ br}^3$$

Öd/ $x^2+y^2 = 4-z$ paraboloidi ile $3z^2 = x^2+y^2$ koni yüzeylerinin sınırladığı cismin hacmini bulunuz.