Toplam Diferansiyel

$z=f(x,y)$ fonksiyonu ve birinci mertebeden $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ kısmi türevleri bir (a,b) noktasının civarında tanımlı ve sürekli olsunlar. Bu fonksiyona ortalama değer teoremini uygulayalım:

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$= f'_x(a+\theta_1\Delta x, b)\cdot\Delta x + f'_y(a+\Delta x, b+\theta_2\Delta y)\cdot\Delta y$$

$\frac{\partial z}{\partial x}=f'_x$ ve $\frac{\partial z}{\partial y}=f'_y$ türevleri (a,b) noktasında sürekli olduklarından

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_1 = 0 \quad \text{ve} \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_2 = 0$$

olmak üzere

$$f'_x(a+\theta_1\Delta x, b) = f'_x(a,b)+\varepsilon_1$$

$$f'_y(a+\Delta x, b+\theta_2\Delta y) = f'_y(a,b)+\varepsilon_2$$

yazabiliriz.

$$\Delta z = f'_x(a+\theta_1 x, b)\cdot\Delta x + f'_y(a+\Delta x, b+\theta_2\Delta y)\cdot\Delta y$$

$$= \left[f'_x(a,b) + \varepsilon_1\right]\Delta x + \left[f'_y(a,b) + \varepsilon_2\right]\Delta y$$

$$= f'_x(a,b)\cdot\Delta x + f'_y(a,b)\cdot\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$\Delta x \to 0$, $\Delta y \to 0$ yaklaştığında $\Delta z \to 0$ 'a yaklaşır. Dolayısıyla Δz bir sonsuz küçüktür. Bu sonsuz küçüğün asal kısmı olan

$$f'_x(a,b)\,\Delta x + f'_y(a,b)\cdot\Delta y$$

ifadesine $z = f(x,y)$ fonksiyonunun (a,b) noktasındaki toplam diferansiyeli denir ve dz ile gösterilir.

$$dz\Big|_{(a,b)} = f'_x(a,b)\cdot dx + f'_y(a,b)\,dy$$

$$z = f(x,y) = x$$

$$dz = dx = 1\cdot\Delta x + 0\cdot\Delta y$$

$$\Delta x = dx$$

$$u = f(x,y,z) \Rightarrow du = f'_x\,dx + f'_y\,dy + f'_z\,dz$$

Ör/ $u = x^2 + y^2 + z^2 \Rightarrow du = ?$

$$du = 2x\,dx + 2y\,dy + 2z\,dz$$

Ör/ $z = \sin 2x - \cos(xy) \Rightarrow dz$

$$dz = [2\cos 2x + y \cdot \sin(xy)]\,dx + x \cdot \sin(xy)\,dy$$

Ör/ $f(x,y,z) = x^3 + y^3 - 3xyz \Rightarrow df(1,1,-2) = ?$

$$df = (3x^2 - 3yz)\,dx + (3y^2 - 3xz)\,dy - 3xy\,dz$$

$$df\Big|_{(1,1,-2)} = 9\,dx + 9\,dy - 3\,dz$$

Bileşik Fonksiyonların Toplam Diferansiyeli

$z = f(x,y) \qquad x = x(s,t) \qquad y = y(s,t)$

$$dz = \frac{\partial f}{\partial x} \cdot \boxed{dx} + \frac{\partial f}{\partial y} \cdot \boxed{dy} \qquad dx = \frac{\partial x}{\partial s} \cdot ds + \frac{\partial x}{\partial t} \cdot dt \qquad dy = \frac{\partial y}{\partial s} \cdot ds + \frac{\partial y}{\partial t} \cdot dt$$

$$dz = \frac{\partial f}{\partial x}\left[\frac{\partial x}{\partial s}\cdot ds + \frac{\partial x}{\partial t}\cdot dt\right] + \frac{\partial f}{\partial y}\left[\frac{\partial y}{\partial s}\, ds + \frac{\partial y}{\partial t}\, dt\right]$$

$$= \underbrace{\left[\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\right]}_{\frac{\partial f}{\partial s}}\cdot ds + \underbrace{\left[\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right]}_{\frac{\partial f}{\partial t}}\cdot dt$$

$$= \frac{\partial f}{\partial s}\, ds + \frac{\partial f}{\partial t}\, dt$$

Ör/

$$\left.\begin{aligned} u &= x^2+y^2+z^2 \\ x &= r\cos t \\ y &= r\sin t \\ z &= t \end{aligned}\right\}\quad du = ?$$

$$du = \frac{\partial u}{\partial r}\, dr + \frac{\partial u}{\partial t}\cdot dt$$

$$= \left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}\right] dr + \left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}\right] dt$$

$$= \left[2x\cdot\cos t + 2y\cdot\sin t + 2z\cdot 0\right] dr + \left[2x\cdot(-r\sin t) + 2y(r\cos t) + 2z\cdot 1\right] dt$$

$$= \left[2r\cos^2 t + 2r\sin^2 t\right] dr + \left[-2r^2\sin t\cos t + 2r^2\sin t\cos t + 2t\right] dt$$

$$= 2r\, dr + 2t\, dt$$

Ör/

$$\left.\begin{array}{l} z=\sin(x^2y) \\ x=st^2 \\ y=s^2+\dfrac{1}{t} \end{array}\right\} \quad dz=?$$

$$dz=\frac{\partial z}{\partial s}\cdot ds+\frac{\partial z}{\partial t}\cdot dt$$

$$=\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\right]\cdot ds+\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right]\cdot dt$$

$$=\left[2xy\cos(x^2y)\cdot t^2+x^2\cos(x^2y)\cdot 2s\right]ds+\left[2xy\cos(x^2y)\cdot 2st+x^2\cos(x^2y)\cdot\left(-\frac{1}{t^2}\right)\right]\cdot dt$$

$$=\left\{\left[2st^2\left(s^2+\frac{1}{t}\right)\cdot\cos\left[s^2t^4\left(s^2+\frac{1}{t}\right)\right]\right]\cdot t^2+s^2t^4\cdot\cos\left[s^2t^4\left(s^2+\frac{1}{t}\right)\right]\cdot 2s\right\}ds$$

$$+\left\{\left[2st^2\left(s^2+\frac{1}{t}\right)\cos\left[s^2t^4\left(s^2+\frac{1}{t}\right)\right]\right]2st+s^2t^4\cos\left[s^2t^4\left(s^2+\frac{1}{t}\right)\right]\cdot\left(-\frac{1}{t^2}\right)\right\}dt$$

Ö5/ $T(x,y,z,t) = \dfrac{xy}{1+z}(1+t)$

$x = t$

$y = 2t$

$z = t - t^2$

$dT = ?$

$$dT = \frac{\partial T}{\partial t} \cdot dt$$

$$= \left[\frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial T}{\partial t} \cdot 1 \right] \cdot dt$$

$$= \left[\frac{y(1+t)}{1+z} \cdot 1 + \frac{x(1+t)}{1+z} \cdot 2 + \left(\frac{-xy(1+t)}{(1+z)^2} \right) \cdot (1-2t) + \frac{xy}{1+z} \right] \cdot dt$$

$$= \left[\frac{2t(1+t)}{1+t-t^2} + 2\frac{t(1+t)}{1+t-t^2} - \frac{2t^2(1+t)}{(1+t-t^2)^2}(1-2t) + \frac{2t^2}{1+t-t^2} \right] dt$$

Yüksek Mertebeden Toplam Diferansiyeller

$z = f(x,y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$d^2 z = \left[\frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y \partial x} dy \right] \cdot dx + \left[\frac{\partial^2 z}{\partial x \partial y} \cdot dx + \frac{\partial^2 z}{\partial y^2} \cdot dy \right] dy = \frac{\partial^2 z}{\partial x^2} \cdot (dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} \cdot (dy)^2$$

$$d^3z = \left[\frac{\partial^3 z}{\partial x^3}\cdot dx + \frac{\partial^3 z}{\partial y \partial x^2}\, dy\right]\cdot (dx)^2 + 2\cdot\left[\frac{\partial^3 z}{\partial x^2 \partial y}\, dx + \frac{\partial^3 z}{\partial y \partial x \partial y}\, dy\right] dx\, dy + \left[\frac{\partial^3 z}{\partial x \partial y^2}\, dx + \frac{\partial^3 z}{\partial y^3}\, dy\right]\cdot (dy)^2$$

$$\vdots$$

$$d^n z = \left(\frac{\partial}{\partial x}\, dx + \frac{\partial}{\partial y}\, dy\right)^{(n)} z$$

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$d^3 z = \frac{\partial^3 z}{\partial x^3}(dx)^3 + 3\frac{\partial^3 z}{\partial y \partial x^2}(dx)^2\, dy + 3\frac{\partial^3 z}{\partial y^2 \partial x}\, dx\, (dy)^2 + \frac{\partial^3 z}{\partial y^3}\cdot (dy)^3$$

$$d^4 z = \frac{\partial^4 z}{\partial x^4}(dx)^4 + 4\frac{\partial^4 z}{\partial y \partial x^3}(dx)^3\cdot dy + 6\frac{\partial^4 z}{\partial y^2 \partial x^2}(dx)^2 (dy)^2 + 4\frac{\partial^4 z}{\partial y^3 \partial x}(dx)(dy)^3 + \frac{\partial^4 z}{\partial y^4}(dy)^4$$