Homojen Olmayan Sınır Şartlarıyla İlgili Problemler

$$\left.\begin{aligned} Ly &= \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=F(x) \qquad \alpha<x<\beta \\ B_1y &:= -l_1y'(\alpha)+h_1y(\alpha)=m_1 \\ B_2y &:= l_2y'(\beta)+h_2y(\beta)=m_2 \end{aligned}\right\} \quad (38)$$

şeklinde bir problemimiz olduğunu varsayalım.

Bu tip bir problemi ya süperpozisyon yöntemi ile ya da Green özdeşliği yöntemi ile çözebiliriz. Her iki yöntemin çözümünde

$$\left.\begin{aligned} Ly &= \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=F(x) \qquad \alpha<x<\beta \\ B_1y &= 0 \\ B_2y &= 0 \end{aligned}\right\} \quad (39)$$

problemi için Green fonksiyonu gözönüne alınarak çözüm bulunur.

1. Süperpozisyon Yöntemi

Bu yöntemde $g(x;T)$ (39) probleminin Green fonksiyonu olmak üzere

$$y_1(x) = \int_{\alpha}^{\beta} g(x,T)\, F(T)\, dT$$

ve y_2'de

$$\left.\begin{aligned} Ly &= 0 \\ B_1 y &= m_1 \\ B_2 y &= m_2 \end{aligned}\right\} (40)$$

probleminin çözümü olmak üzere (38) probleminin çözümü $y = y_1 + y_2$ şeklinde hesaplanır.

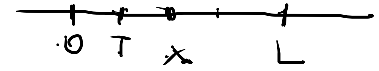
Ör/ $-z \dfrac{d^2 y}{dx^2} = F(x) \qquad 0 < x < L$

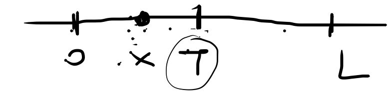
$y(0) = m_1$

$y(L) = m_2$

sınır değer-probleminin çözümünü bulunuz.

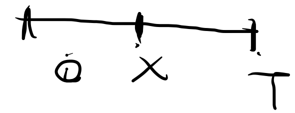
$$\left.\begin{aligned} -\tau \frac{d^2y}{dx^2} &= F(x) \quad 0<x<L \\ y(0)&=0 \\ y(L)&=0 \end{aligned}\right\}$$ problemının Green fonksiyonunu

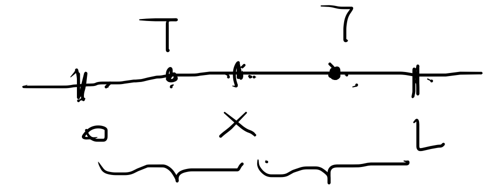




$$g(x;T) = \begin{cases} \dfrac{x(L-T)}{L\tau} & 0 \leq x \leq T \\ \dfrac{T(L-x)}{L\tau} & T \leq x \leq L \end{cases}$$

şeklinde daha önce bulmuştuk.





$$y_1(x) = \int_0^L g(x;T)\,F(T)\,dT$$

$$= \int_0^x \frac{T(L-x)}{L\tau} F(T)\,dT + \int_x^L \frac{x(L-T)}{L\tau} F(T)\,dT$$

$$= \frac{L-x}{L\tau} \int_0^x T F(T)\,dT + \frac{x}{L\tau} \int_x^L (L-T) F(T)\,dT$$

$$\left.\begin{aligned} -\tau \frac{d^2y}{dx^2} &= 0 \\ y(0)&=m_1 \\ y(L)&=m_2 \end{aligned}\right\}$$ problemini çözelim;

$y'' = 0 \Rightarrow \overset{\text{k.D.}}{r^2} = 0 \Rightarrow r_{1,2} = 0$

$y = Ax + B$

$y(0) = m_1 \Rightarrow B = m_1$

$y(L) = m_2 \Rightarrow AL + m_1 = m_2 \Rightarrow A = \dfrac{m_2 - m_1}{L}$

$$y_2 = \frac{m_2 - m_1}{L} x + m_1$$

$$y = y_1 + y_2 = \frac{L-x}{L\sigma}\int_0^x T F(T)\,dT + \frac{x}{L\sigma}\int_x^L (L-T)F(T)\,dT + \frac{m_2-m_1}{L}x + m_1$$

2. Green Özdeşliği Yöntemi

(38) probleminin çözümü istendiğinde $\int_\alpha^\beta (uLv - vLu)\,dx = p(x).\left[u v' - v u'\right]\Big|_\alpha^\beta$ Green özdeşliğinde

$u = y(x)$ $\quad v = g(x,t)$ (Green fonksiyonu) olarak alınırsa

$$\int_\alpha^\beta (yLg - gLy)\,dx = p(x).\left[y.\frac{\partial g}{\partial x} - g.\frac{\partial y}{\partial x}\right]_\alpha^\beta \quad \text{olur.}$$

$$\int \delta(x-x_0).f(x)\,dx = f(x_0)$$

$Ly = F(x)$ ve $Lg = \delta(x-T)$ olduğundan

$$\int_\alpha^\beta \left[y.\delta(x-T) - g.F(x)\right]dx = p(\beta)\left[y(\beta).\frac{\partial g(\beta;T)}{\partial x} - g(\beta;T).\frac{\partial y(\beta)}{\partial x}\right] - p(\alpha)\left[y(\alpha).\frac{\partial g(\alpha;T)}{\partial x} - g(\alpha;T)\frac{\partial y(\alpha)}{\partial x}\right]$$

$$y(T) - \int_\alpha^\beta g(x;T)F(x)\,dx = p(\beta)\left[y(\beta).\frac{\partial g(\beta;T)}{\partial x} - g(\beta;T).\frac{\partial y(\beta)}{\partial x}\right] - p(\alpha)\left[y(\alpha).\frac{\partial g(\alpha;T)}{\partial x} - g(\alpha;T)\frac{\partial y(\alpha)}{\partial x}\right]$$

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = m_1 \Rightarrow y'(\alpha) = \frac{m_1 - h_1 y(\alpha)}{-l_1} \Rightarrow y'(\alpha) = -\frac{m_1}{l_1} + \frac{h_1}{l_1} y(\alpha)$$

$$B_2 y = l_2 y'(\beta) + h_2 y(\beta) = m_2 \Rightarrow y'(\beta) = \frac{m_2 - h_2 y(\beta)}{l_2} \Rightarrow y'(\beta) = \frac{m_2}{l_2} - \frac{h_2}{l_2} y(\beta)$$

$l_1 = l_2 = 0$

$B_1 y = h_1 y(\alpha) = m_1 \Rightarrow y(\alpha) = \frac{m_1}{h_1}$

$B_2 y = h_2 y(\beta) = m_2 \Rightarrow y(\beta) = \frac{m_2}{h_2}$

$$Lg = \delta(x-T)$$
$$B_1 g = 0$$
$$B_2 g = 0$$
$$h_2 g(\beta;T) = 0$$
$$h_1 g(\alpha;T) = 0$$

$$= p(\beta)\left[\frac{m_2}{h_2}\cdot\frac{\partial g(\beta;T)}{\partial x} - g(\beta;T)\cdot y'(\beta)\right]$$

$$y(T) - \int_\alpha^\beta g(x;T)F(x)\,dx = p(\beta)\left[y(\beta)\frac{\partial g(\beta;T)}{\partial x} - g(\beta;T)\cdot\left(\frac{m_2}{l_2} - \frac{h_2}{l_2}y(\beta)\right)\right]$$

$$-p(\alpha)\left[\frac{m_1}{h_1}\frac{\partial g(\beta;T)}{\partial x} - g(\beta;T)\cdot y'(\alpha)\right]$$

$$-p(\alpha)\left[y(\alpha)\frac{\partial g(\alpha;T)}{\partial x} - g(\alpha;T)\cdot\left(-\frac{m_1}{l_1} + \frac{h_1}{l_1}y(\alpha)\right)\right]$$

$$p(\beta)\frac{m_2}{h_2}\left[\frac{\partial g(\beta;T)}{\partial x} - \frac{h_2}{m_2}g(\beta;T)y'(\beta)\right] - p(\alpha)\cdot\frac{m_1}{h_1}\left[\frac{\partial g(\alpha;T)}{\partial x} - \frac{h_1}{m_1}g(\alpha;T)y'(\alpha)\right]$$

$$= p(\beta)\left\{y(\beta)\left[\frac{\partial g(\beta;T)}{\partial x} + \frac{h_2}{l_2}g(\beta;T)\right] - \frac{m_2}{l_2}g(\beta;T)\right\}$$

$$p(\beta)\frac{m_2}{h_2}\frac{\partial g(\beta;T)}{\partial x} - p(\alpha)\frac{m_1}{h_1}\frac{\partial g(\alpha;T)}{\partial x}$$

$$-p(\alpha)\left\{y(\alpha)\left[\frac{\partial g(\alpha;T)}{\partial x} - \frac{h_1}{l_1}g(\alpha;T)\right] + \frac{m_1}{l_1}g(\alpha;T)\right\}$$

$$y(T) = \int_\alpha^\beta g(x;T)F(x)\,dx + p(\beta)m_2\frac{\partial g(\beta;T)}{\partial x} - p(\alpha)m_1\frac{\partial g(\alpha;T)}{\partial x}$$

$$= p(\beta)\left\{\frac{+y(\beta)}{l_1}\left[\underbrace{l_2\frac{\partial g(\beta;T)}{\partial x} + h_2 g(\beta;T)}_{=0}\right] - \frac{m_2}{l_2}g(\beta;T)\right\}$$

$$y(x) = \int_\alpha^\beta g(x;T)F(T)\,dT + p(\beta)m_2\frac{\partial g(x;\beta)}{\partial T} - p(\alpha)m_1\frac{\partial g(x;\alpha)}{\partial T}$$

$$-p(\alpha)\left\{\frac{-y(\alpha)}{l_1}\left[\underbrace{l_1\frac{\partial g(\alpha;T)}{\partial x} + h_1 g(\alpha;T)}_{=0}\right] + \frac{m_1}{l_1}g(\alpha;T)\right\}$$

$$\Rightarrow\quad y(T) = \int_\alpha^\beta g(x;T)F(x)\,dx - p(\alpha)\frac{m_1}{l_1}g(\alpha;T) - p(\beta)\frac{m_2}{l_2}g(\beta;T)$$

$$(l_1 \neq 0,\ l_2 \neq 0)$$

$$y(x) = \int_\alpha^\beta g(T;x)F(T)\,dT - p(\alpha)\cdot$$

x ile T'yi yer değiştirelim

$$\Rightarrow y(x) = \int_{\alpha}^{\beta} g(x;T)\,F(T)\,dT - p(\alpha)\frac{m_1}{l_1}\,g(x;\alpha) - p(\beta)\frac{m_2}{l_2}\,g(x;\beta) \qquad (l_1 \neq 0,\ l_2 \neq 0) \qquad (41)$$

şeklinde çözüm elde edilir.

Eğer $l_1 = l_2 = 0$ ise ve $h_1 = h_2 = 1$ ise

$$y(x) = \int_{\alpha}^{\beta} g(x;T)\,F(T)\,dT - m_1\,p(\alpha)\frac{\partial g(x;\alpha)}{\partial T} + m_2\,p(\beta)\frac{\partial g(x;\beta)}{\partial T} \qquad (42)$$

şeklinde çözüm belirlenir. $y(0) = m_1 \quad y(L) = m_2 \quad (l_1 = l_2 = 0)$

Ör/ Aynı problemi alalım:

$$g(x;T) = \frac{1}{L\tau}\left[x\underbrace{(L-T)\,H(T-x)}_{} + T\underbrace{(L-x)}_{}H(x-T)\right]$$

$x(L-T) \quad T>x$ / $0 \quad T<x$ ($x<T<L$) $\qquad$ $T(L-x) \quad x>T$ / $0 \quad x<T$ ($0<T<x$)

$$y(x) = \int_0^x T\cdot\frac{(L-x)}{L\tau}F(T)\,dT + \int_x^L x\frac{(L-T)}{L\tau}F(T)\,dT + m_2\cdot(-\tau)\cdot\frac{(-x)}{L\tau} - m_1\cdot\frac{(L-x)}{L\tau}\cdot(-\tau)$$

$$= \int_0^x T\frac{(L-x)}{L\tau}F(T)\,dT + \int_x^L \frac{x(L-T)}{L\tau}F(T)\,dT + \underbrace{m_2\frac{x}{L} + m_1\frac{L-x}{L}}_{(m_2-m_1)\frac{x}{L}+m_1}$$