2) Homojen Diferansiyel Denklemler

Diferansiyel denklem $\frac{dy}{dx}=f(x,y)$ şeklinde yazıldığında $f(x,y)=g\left(\frac{y}{x}\right)$ şeklinde ifade edilebiliyorsa denkleme homojen dif. denklem denir.

Veya dif. denklem $P(x,y)\,dx+Q(x,y)\,dy=0$ şeklinde verildiğinde P ve Q fonksiyonları aynı dereceden homojen fonksiyonlar ise denkleme homojen dif. denklem denir.

Homojen fonk:

$f(x,y)$ fonksiyonu için $f(\lambda x,\lambda y)=\lambda^n f(x,y)$ oluyorsa $f(x,y)$ x ve y'ye göre n. dereceden homojen fonksiyondur denir.

Çözüm için $\frac{y}{x}=u$ dönüşümü yapılır. Dolayısıyla $y=u.x \Rightarrow y'=u'x+u$
$(dy=u\,dx+x\,du)$

Ör/ $(x^3+y^3)\,dx - xy^2\,dy = 0$ dif. denk'nin genel çözümünü bulunuz.

$P(x,y) = x^3+y^3$ $\quad P(\lambda x,\lambda y) = (\lambda x)^3 + (\lambda y)^3 = \lambda^3 x^3 + \lambda^3 y^3 = \lambda^3(x^3+y^3)$

$= \lambda^{3} P(x,y)$

3. dereceden homojen

$Q(x,y) = -xy^2$ $\quad Q(\lambda x,\lambda y) = -(\lambda x)(\lambda y)^2 = -\lambda x.\lambda^2 y^2 = -\lambda^3 xy^2 = \lambda^3(-xy^2)$

$= \lambda^{3} Q(x,y)$

3. dereceden homojen

Dif. denk. homojen dif. denklemdir.

$\frac{y}{x} = u \Rightarrow y = ux \Rightarrow dy = u\,dx + x\,du$

$\Rightarrow [x^3 + (ux)^3]\,dx - x.(ux)^2\,[u\,dx + x\,du] = 0$

$\Rightarrow \underbrace{x^3(1+u^3)\,dx - u^3x^3\,dx}_{x^3(1+\cancel{u^3}-\cancel{u^3})dx} - x^4u^2\,du = 0 \Rightarrow \frac{\cancel{x^3}\,dx}{x^{\cancel{4}}} - \frac{\cancel{x^4}\,u^2\,du}{\cancel{x^4}} = \frac{0}{x^4}$ $\quad (x \neq 0)$

$$\int \frac{dx}{x} - \int u^2 du = \int 0 \Rightarrow \ln|x| - \frac{u^3}{3} = c \Rightarrow \boxed{\ln|x| - \frac{y^3}{3x^3} = C} \quad \text{G.Ç.}$$

Ör/ $\left(x.\cos\frac{y}{x}\right)\frac{dy}{dx} = \left(y.\cos\frac{y}{x}\right) + x$ dif. denk'nin genel çözümünü bulunuz.

$$\frac{dy}{dx} = \frac{\left(y.\cos\frac{y}{x}\right) + x}{x.\cos\frac{y}{x}} \qquad \left(x.\cos\frac{y}{x} \neq 0\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{\cos\frac{y}{x}} = g\left(\frac{y}{x}\right)$$ Homojen dif. denk.

$$\frac{y}{x} = u \Rightarrow y = u.x \Rightarrow y' = u'x + u \Rightarrow u'x + \cancel{u} = \cancel{u} + \frac{1}{\cos u} \Rightarrow u'x = \frac{1}{\cos u}$$

$$u'x = \frac{1}{\cos u} \Rightarrow \frac{du}{dx}\cdot x = \frac{1}{\cos u} \Rightarrow \int \cos u \, du = \int \frac{dx}{x} \qquad (x \neq 0)$$

$$\sin u = \ln|x| + \ln c$$

$$\sin u = \ln(cx)$$

$$\boxed{\sin\frac{y}{x} = \ln(cx)} \Rightarrow \frac{y}{x} = \arcsin[\ln(cx)]$$

$$\downarrow$$

$$F(x,y,c)=0 \qquad \Rightarrow \boxed{y = x\cdot\arcsin[\ln(cx)]}$$

$$\downarrow$$

$$y = f(x,c)$$

Ör/ $\left(1+2e^{x/y}\right)dx + 2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

dif. denk.'nin genel çözümünü bulunuz.

$$\Rightarrow \frac{dx}{dy} = \frac{-2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+2e^{x/y}} \quad \text{Homojen d.d.}$$

$$= g\left(\frac{x}{y}\right)$$

$$P(x,y) = 1+2e^{x/y} \Rightarrow P(\lambda x, \lambda y) = 1+2e^{\frac{\lambda x}{\lambda y}}$$

$$= 1+2e^{x/y} \rightarrow \text{0. dereceden homojen fonk.}$$

$$Q(x,y)=2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right) \Rightarrow Q(\lambda x,\lambda y)=2e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)=2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=Q(x,y)$$

0. dereceden homojen fonk.

$$\frac{x}{y}=u \Rightarrow x=u\cdot y \Rightarrow \frac{dx}{dy}=u'y+u$$

$$u'y+u=\frac{-2e^{u}(1-u)}{1+2e^{u}} \Rightarrow u'y=\frac{-2e^{u}(1-u)}{1+2e^{u}}-u$$

$$\Rightarrow y\cdot\frac{du}{dy}=\frac{-2e^{u}+\cancel{2ue^{u}}-u-\cancel{2ue^{u}}}{1+2e^{u}}$$

$$\Rightarrow y\cdot\frac{du}{dy}=\frac{-2e^{u}-u}{1+2e^{u}} \Rightarrow \int\frac{(1+2e^{u})}{u+2e^{u}}\,du=-\int\frac{dy}{y}$$

$$\Rightarrow \ln|u+2e^{u}|=-\ln|y|+\ln c$$

$$u + 2e^{u} = \frac{c}{y}$$

$$\frac{x}{y} + 2e^{\frac{x}{y}} = \frac{c}{y} \Rightarrow \boxed{x + 2ye^{\frac{x}{y}} = C} \quad \text{G.Ç.}$$

Ör/ $\left(x + \sqrt{y^2 - xy}\right)\frac{dy}{dx} = y$ dif. denkleminin $y\left(\frac{1}{2}\right) = 1$ koşuluna uygun çözümünü bulunuz.

$$y\,dx - \left(x + \sqrt{y^2 - xy}\right)dy = 0$$

$P(x,y) = y \Rightarrow P(\lambda x, \lambda y) = \lambda y = \lambda P(x,y) \rightarrow$ 1. dereceden homojen fonk.

$$Q(x,y) = -\left(x + \sqrt{y^2 - xy}\right) \Rightarrow Q(\lambda x, \lambda y) = -\left[\lambda x + \sqrt{(\lambda y)^2 - \lambda x \lambda y}\right]$$

$$= -\left[\lambda x + \sqrt{\lambda^2 (y^2 - xy)}\right]$$

$$= \lambda\left[-\left(x + \sqrt{y^2 - xy}\right)\right] = \lambda . Q(x,y) \rightarrow$$ 1. dereceden homojen fonk.

Denklem homojendir.

$$\frac{y}{x}=u \Rightarrow y=ux \Rightarrow dy=u\,dx+x\,du$$

$$ux\,dx-\left[x+\sqrt{u^2x^2-x\cdot ux}\right](u\,dx+x\,du)=0$$

$$\left[\cancel{ux}-\cancel{ux}-u\cdot\sqrt{u^2x^2-ux^2}\right]dx-\left[x^2+x\sqrt{u^2x^2-ux^2}\right]du=0$$

$$\frac{\left(-\cancel{u}\cancel{x}\sqrt{\cancel{u^2-u}}\right)dx}{x^{\cancel{2}}\,\cancel{u}\cdot\sqrt{\cancel{u^2-u}}}-\frac{\cancel{x^2}\left[1+\sqrt{u^2-u}\right]du}{\cancel{x^2}u\sqrt{u^2-u}}=\frac{0}{x^2u\sqrt{u^2-u}}$$

$(x\neq 0, u\neq 0)$

$(u\neq 1)$

$$\Rightarrow -\frac{dx}{x}-\frac{1+\sqrt{u^2-u}}{u\sqrt{u^2-u}}\,du=0 \Rightarrow \frac{dx}{x}+\left[\frac{1}{u\sqrt{u}\sqrt{u-1}}+\frac{1}{u}\right]du=0$$

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$$\frac{du}{u\sqrt{u}\sqrt{u-1}} \qquad u=t^2$$