Ters Laplace Dönüşümü

$\mathcal{L}\{f(t)\} = F(s)$

$f(t)$'ye $F(s)$'in ters Laplace dönüşümü denir.

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

şeklinde gösterilir.

- $F(s) = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 = f(t)$

  $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$

  $F(s) = \frac{1}{s^3} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!} = f(t)$

  $\vdots$

  $F(s) = \frac{1}{s^{n+1}} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!} = f(t)$

- $F(s) = \frac{1}{s-a} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} = f(t)$

  $F(s) = \frac{1}{(s-a)^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at} = f(t)$

  $F(s) = \frac{1}{(s-a)^3} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^3}\right\} = \frac{t^2}{2!}e^{at} = f(t)$

  $\vdots$

  $F(s) = \frac{1}{(s-a)^{n+1}} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = \frac{t^n}{n!}e^{at} = f(t)$

- $F(s) = \frac{1}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

- $F(s) = \frac{1}{s^2-a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

- $F(s) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

- $F(s) = \frac{s}{s^2-a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Ters Laplace Dönüşümünün Özellikleri

1°) Lineerlik özelliği

$$\mathcal{L}^{-1}\{c_1F_1(s) \mp c_2F_2(s) \mp \cdots\} = c_1\mathcal{L}^{-1}\{F_1(s)\} \mp c_2\mathcal{L}^{-1}\{F_2(s)\} + \cdots$$

Ör/ $$\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 4 . e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

2°) kaydırma özelliği

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \Rightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{-at} f(t)$$

Ör/ $$\mathcal{L}^{-1}\left\{\frac{1}{\underbrace{s^2-2s+5}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\underbrace{s^2-2s+1}+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = e^{-t} . \frac{\sin 2t}{2}$$

Örnekler

1) $\mathcal{L}\{t^3 . e^{-3t}\} = ?$

$\mathcal{L}\{t^n f(t)\} = (-1)^n . \frac{d^n}{ds^n} . F(s)$

$f(t) = e^{-3t}$

$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$

$\frac{d}{ds}\left(\frac{1}{s+3}\right) = \frac{-1}{(s+3)^2} \Rightarrow \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{2}{(s+3)^3}$

$\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = \frac{d}{ds}\left(\frac{2}{(s+3)^3}\right) = \frac{-6}{(s+3)^4}$

$\Rightarrow \mathcal{L}\{t^3 . e^{-3t}\} = (-1)^3 . \left(\frac{-6}{(s+3)^4}\right) = \frac{6}{(s+3)^4}$

2) $\mathcal{L}\{(t+2)^2 e^t\} = \mathcal{L}\{(t^2+4t+4)e^t\} = \mathcal{L}\{t^2 e^t\} + 4\mathcal{L}\{te^t\} + 4\mathcal{L}\{e^t\}$

$\mathcal{L}\{e^t\} = \frac{1}{s-1}$

$= (-1)^2 . \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) + 4 . \frac{1}{(s-1)^2} + 4 \frac{1}{s-1}$

$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}$

3) $\mathcal{L}\{(t^2-1)^2 \sinh 3t\} = \mathcal{L}\left\{(t^4-2t^2+1)\cdot \frac{e^{3t}-e^{-3t}}{2}\right\} = \frac{1}{2}\mathcal{L}\{t^4e^{3t}-2t^2e^{3t}+e^{3t}-t^4e^{-3t}+2t^2e^{-3t}-e^{-3t}\}$

$\sinh 3t = \frac{e^{3t}-e^{-3t}}{2}$

$= \frac{1}{2}\mathcal{L}\{t^4e^{3t}\} - \mathcal{L}\{t^2e^{3t}\} + \frac{1}{2}\mathcal{L}\{e^{3t}\} - \frac{1}{2}\mathcal{L}\{t^4e^{-3t}\} + \mathcal{L}\{t^2e^{-3t}\} - \frac{1}{2}\mathcal{L}\{e^{-3t}\}$

$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$

$= \frac{1}{2}(-1)^4\cdot\frac{d^4}{ds^4}\left(\frac{1}{s-3}\right) - (-1)^2\cdot\frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right) - \frac{1}{2}(-1)^4\frac{d^4}{ds^4}\left(\frac{1}{s+3}\right) + (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) - \frac{1}{2}\left(\frac{1}{s+3}\right)$

$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$

$= \frac{1}{2}\left(\frac{24}{(s-3)^5}\right) - \left(\frac{2}{(s-3)^2}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right) - \frac{1}{2}\left(\frac{24}{(s+3)^5}\right) + \frac{2}{(s+3)^2} - \frac{1}{2}\left(\frac{1}{s+3}\right)$

4) $\mathcal{L}\{e^t\cdot\cos^2 t\} = \mathcal{L}\left\{e^t\cdot\frac{(1+\cos 2t)}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{e^t\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s-1}\right] + \frac{1}{2}\left[\frac{s-1}{(s-1)^2+4}\right]$

5) $f(t) = t\cdot\cos 2t$ $\qquad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\{f''(t)\} = ?$

$f(t) = t\cos 2t \rightarrow f(0) = 0$

$f'(t) = \cos 2t - 2t\sin 2t \rightarrow f'(0) = 1$

$$\mathcal{L}\{t.\cos 2t\} = (-1) \cdot \frac{d}{ds}\left(\frac{s}{s^2+4}\right) = (-1) \cdot \left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right] = \frac{s^2-4}{(s^2+4)^2} = F(s)$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \left(\frac{s^2-4}{(s^2+4)^2}\right) - s \cdot 0 - 1 = s^2\left[\frac{s^2-4}{(s^2+4)^2}\right] - 1$$

6) $F(s) = \dfrac{3}{s+4} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{3}{s+4}\right\} = 3\mathcal{L}^{-1}\left\{\dfrac{1}{s+4}\right\} = 3 \cdot e^{-4t}$

7) $F(s) = \dfrac{1}{2s-5} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{2(s-\frac{5}{2})}\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s-\frac{5}{2}}\right\} = \dfrac{1}{2}e^{+\frac{5t}{2}}$

8) $F(s) = \dfrac{8s}{s^2+16} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{8s}{s^2+16}\right\} = 8\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+16}\right\} = 8\cos 4t$

9) $F(s) = \dfrac{6}{s^2-9} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{6}{s^2-9}\right\} = 6\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-9}\right\} = \dfrac{6}{3}\sinh 3t = 2\sinh 3t$

10) $F(s)=\frac{1}{s^{10}} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^{10}}\right\}=\frac{t^9}{9!}$

11) $F(s)=\frac{2s+1}{s^2+s} \Rightarrow \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+s}\right\}=\mathcal{L}^{-1}\left\{\frac{2\cancel{s}}{\cancel{s}(s+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+s}\right\}$ 1. yol

$=2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2-\frac{1}{4}}\right\}$

$=2.e^{-t}+e^{-t/2}.\frac{\sinh\frac{t}{2}}{1/2}$

$=2e^{-t}+2e^{-t/2}\sinh\frac{t}{2}=2e^{-t}+2e^{-t/2}.\left[\frac{e^{t/2}-e^{-t/2}}{2}\right]$

$=2e^{-t}+1-e^{-t}$

$=e^{-t}+1$

2. yol

$\frac{2s+1}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$

$A=1$

$B=1$

$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+s}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s+1}\right\}$

$=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$=1+e^{-t}$

12) $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}=\mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}=6\mathcal{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+16}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$

$=6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\}+12\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 12\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 8\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6.e^{2t}.\cos 4t + 8e^{2t}\sin 4t$$

13) $$\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{s+3^{+1-1}}{(s+4)^2}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{\cancel{s+4}}{(s+4)^{\cancel{2}}}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4.e^{-4t} - 4te^{-4t}$$

14) $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+2} - \frac{1}{s+1}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 2.e^{-2t} - e^{-t}$$

$$\frac{s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$A = 2$

$B = -1$

15) $\mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2+1}{(s-2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{\cancel{s-2}}{(s-2)^{\cancel{4}}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}$$

$$= e^{2t}\cdot\frac{t^2}{2!} + e^{2t}\cdot\frac{t^3}{3!}$$

16) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{a^2}\cdot\frac{1}{s^2} + \left(-\frac{1}{a^2}\right)\left(\frac{1}{s^2+a^2}\right)\right\} = \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}$

$$= \frac{t}{a^2} - \frac{1}{a^2}\cdot\frac{\sin at}{a}$$

$$= \frac{at - \sin at}{a^3}$$

$$\frac{1}{s^2(s^2+a^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+a^2}$$

$$1 = As(s^2+a^2) + B(s^2+a^2) + (Cs+D)s^2$$

$$1 \equiv As^3 + Asa^2 + Bs^2 + Ba^2 + Cs^3 + Ds^2$$

$$(A+C)s^3 + (B+D)s^2 + Asa^2 + Ba^2 \equiv 1$$

$A+C=0$

$B+D=0 \Rightarrow D = -1/a^2$

$A.a^2 = 0 \Rightarrow A = 0 \Rightarrow C = 0$

$Ba^2 = 1 \Rightarrow B = \frac{1}{a^2}$