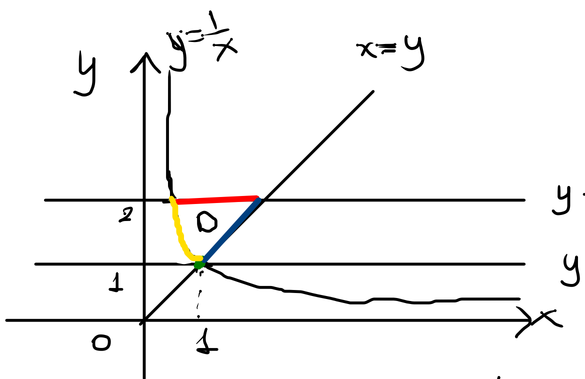
Öd/ $\int_1^2\int_{1/y}^{y} \sqrt{\frac{y}{x}} \cdot e^{\sqrt{x\cdot y}}\, dx\,dy$ integralini uygun bir değişken dönüşümü yaparak hesaplayınız.

$\frac{y}{x}=u^2 \qquad xy=v^2 \qquad \left\{ u=\sqrt{\frac{y}{x}},\ v=\sqrt{xy} \right\} \qquad \frac{y}{x}\cdot xy = u^2v^2 \qquad y=2 \Rightarrow 4=u^2v^2 \Rightarrow u^2=\frac{4}{v^2} \Rightarrow u=\mp\frac{2}{v}$

$y^2=u^2v^2$

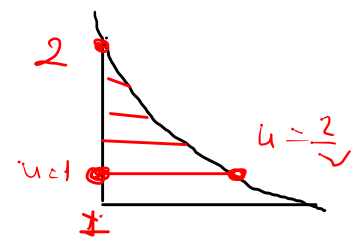
$1\le y\le 2$

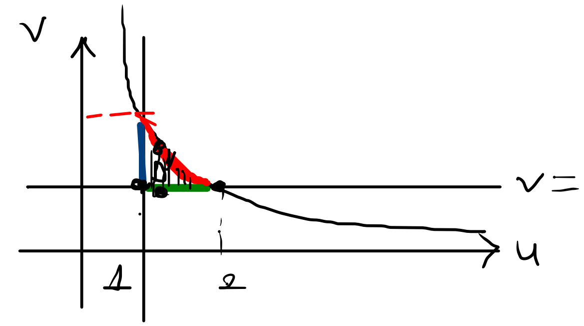
$\frac{1}{y}\le x\le y$



$x=y \Rightarrow u^2=1 \Rightarrow u=\mp 1$

$x=\frac{1}{y} \Rightarrow v^2=1 \Rightarrow v=\mp 1$





$$J=\frac{D(x,y)}{D(u,v)}=\frac{1}{\frac{D(u,v)}{D(x,y)}}=\frac{1}{\begin{vmatrix} \frac{-\frac{y}{x^2}}{2\sqrt{\frac{y}{x}}} & \frac{y}{2\sqrt{xy}} \\ \frac{1/x}{2\sqrt{\frac{y}{x}}} & \frac{x}{2\sqrt{xy}} \end{vmatrix}}$$

$$=\frac{1}{-\frac{y}{2x^{\cancel{2}}\sqrt{\frac{y}{x}}}\cdot\frac{\cancel{x}}{2\sqrt{xy}}-\frac{y}{2x\sqrt{\frac{y}{x}}\cdot 2\sqrt{xy}}}=\frac{1}{-\frac{1}{4x}-\frac{1}{4x}}=-2x$$

$$I=\iint_{D'} f(u,v)\cdot|J|\,dA'$$

$$=\iint_{D'}\sqrt{\frac{y}{x}}\cdot e^{\sqrt{xy}}\cdot 2x\,dA'$$

$$=2\iint_{D'}\sqrt{xy}\cdot e^{\sqrt{xy}}\,dA'$$

$$= 2\iint_{D'} \sqrt{xy} \cdot e^{\sqrt{xy}}\, dA' = 2\int_1^2\int_1^{2/v} v \cdot e^v \, du\, dv = 2\int_1^2 \left[v \cdot e^v \cdot u \Big|_1^{2/v} \right] dv = 2\int_1^2 \left[\cancel{v} e^v \left(\frac{2}{\cancel{v}} - 1 \right) \right] dv$$

$v = t \qquad e^v dv = ds$

$dv = dt \qquad e^v = s$

$$= 2\int_1^2 \left(2e^v - ve^v \right) dv$$

$$= 4\int_1^2 e^v \, dv - 2\int_1^2 ve^v \, dv$$

$$= 4\, e^v \Big|_1^2 - 2\left[v e^v \Big|_1^2 - \int_1^2 e^v dv \right]$$

$$= 4\left(e^2 - e^1\right) - 2\left[\left(2e^2 - e\right) - \left(e^v \Big|_1^2 \right) \right]$$

$$= \cancel{4e^2} - 4e - \cancel{4e^2} + 2e + 2\left(e^2 - e\right)$$

$$= -2e + 2e^2 - 2e$$

$$= 2e^2 - 4e$$

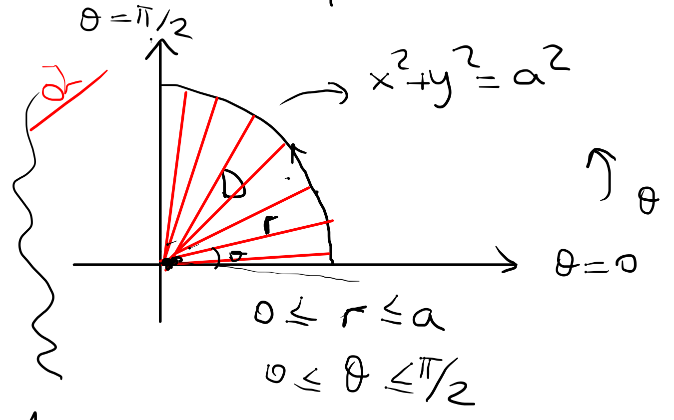
2) Kutupsal Koordinatlarda Değişken Dönüşümü

$I=\iint_D f(x,y)\,dA$ integralinde $x=r\cos\theta$, $y=r\sin\theta$ değişken dönüşümü yapıldığında D bölgesi kutupsal koordinatlara göre taranır, Jakobien hesaplanır ve I integrali

$$\boxed{I=\iint_D f(r\cos\theta, r\sin\theta)\,\underbrace{r}_{J}\,dr\,d\theta}$$

$$J=\frac{D(x,y)}{D(r,\theta)}=\begin{vmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{vmatrix}$$
$$=r\cos^2\theta+r\sin^2\theta$$
$$=r$$

şeklini alır.



Çembersel bölgede dönüşüm

$x=r\cos\theta$
$y=r\sin\theta$
$J=r$

$x^2+y^2=a^2$
$\downarrow$
D bölgesi

Eliptik bölgede dönüşüm

$x=ar\cos\theta$
$y=br\sin\theta$
$J=abr$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
$\downarrow$
D bölgesi

Kutupsal koordinatlar için iki katlı integral ile alan hesaplanmak istenirse;

$$\boxed{A=\iint_D r\,dr\,d\theta \text{ dır.}}$$

Ör: $I = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$ integralini hesaplayalım.

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2} \cdot e^{y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 e^{r^2} \cdot r\,dr\,d\theta = \int_0^{\pi/2}\left[\int_0^1 e^t \cdot \frac{dt}{2}\right]d\theta = \frac{1}{2}\int_0^{\pi/2}\left[e^t \Big|_0^1\right]d\theta$$

$$= \frac{e-1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{e-1}{2}\left(\theta \Big|_0^{\pi/2}\right)$$

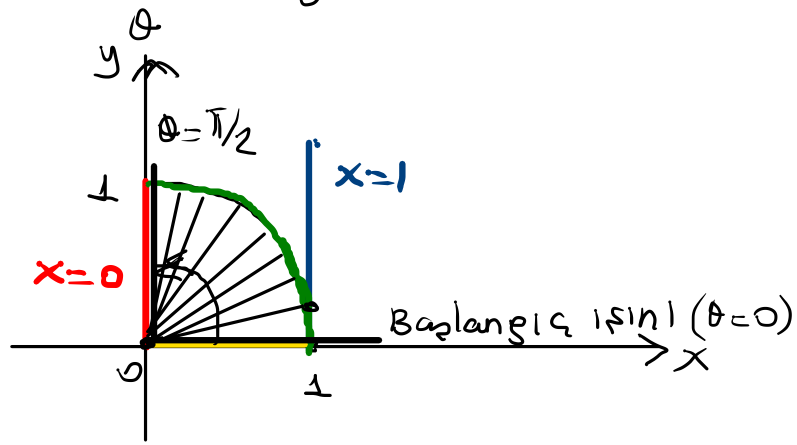
$$= \frac{(e-1)\pi}{4}$$

$0 \leq x \leq 1$

$0 \leq y \leq \underbrace{\sqrt{1-x^2}}$

$\downarrow$

$x^2+y^2=1$



$x = r\cos\theta$

$y = r\sin\theta$

$J = r$

$x^2+y^2=r^2$

$r^2 = t$

$2r\,dr = dt$

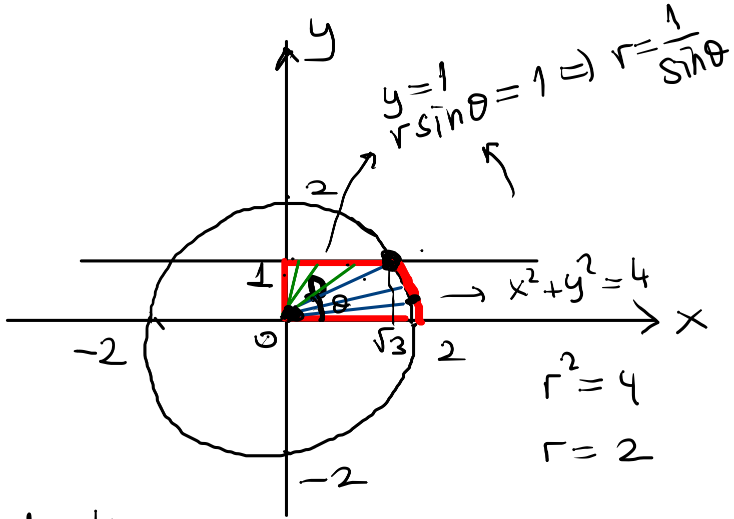
$r\,dr = \frac{dt}{2}$

$r=0 \Rightarrow t=0$

$r=1 \Rightarrow t=1$

Ör/ $x^2+y^2=4$, $y=1$, $x=0$, $y=0$ ile sınırlı bölgenin alanını hesaplayınız.

$x=r\cos\theta$
$y=r\sin\theta$
$J=r$



$y=1 \Rightarrow x^2+1=4$
$x^2=3 \Rightarrow x=\mp\sqrt{3}$

$\tan\theta=\frac{1}{\sqrt{3}}$
$\theta=\frac{\pi}{6}$

$$A=\int_0^{\pi/6}\int_0^{2} r\,dr\,d\theta+\int_{\pi/6}^{\pi/2}\int_0^{1/\sin\theta} r\,dr\,d\theta$$

$$=\int_0^{\pi/6}\left(\frac{r^2}{2}\Big|_0^2\right)d\theta+\int_{\pi/6}^{\pi/2}\left(\frac{r^2}{2}\Big|_0^{1/\sin\theta}\right)d\theta$$

$$=\int_0^{\pi/6}(2-0)\,d\theta+\int_{\pi/6}^{\pi/2}\left(\frac{1}{2\sin^2\theta}\right)d\theta$$

$$=2\theta\Big|_0^{\pi/6}+\frac{1}{2}\int_{\pi/6}^{\pi/2}\csc^2\theta\,d\theta$$

$$=2\left(\frac{\pi}{6}-0\right)+\frac{1}{2}\left[(-\text{cotan}\,\theta)\Big|_{\pi/6}^{\pi/2}\right]$$

$$=\frac{\pi}{3}+\frac{1}{2}\left[-\text{cotan}\frac{\pi}{2}+\text{cotan}\frac{\pi}{6}\right]$$

$$=\frac{\pi}{3}+\frac{1}{2}\left[0+\sqrt{3}\right] \Rightarrow A=\frac{\pi}{3}+\frac{\sqrt{3}}{2}\ \text{br}^2$$

Ör/ $x^2+y^2=2x$ ve $x^2+y^2=2y$ çemberlerinin sınırladığı bölgenin alanını bulunuz.

$x^2-2x+y^2=0$

$(x-1)^2-1+y^2=0$

$(x-1)^2+y^2=1$

$M(1,0)$

$R=1$

$x^2+y^2-2y=0$

$x^2+(y-1)^2-1=0$

$x^2+(y-1)^2=1$

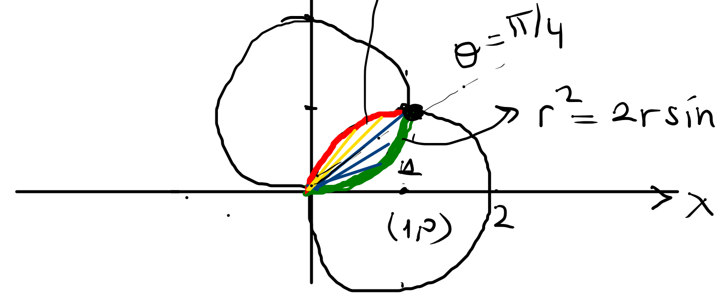
$M(0,1)$

$R=1$

$r^2=2r\cos\theta$

$r(r-2\cos\theta)=0$

$r=2\cos\theta$



$r^2=2r\sin\theta \Rightarrow r\,(r-2\sin\theta)=0$

$r=2\sin\theta$

$x=r\cos\theta$

$y=r\sin\theta$

$J=r$

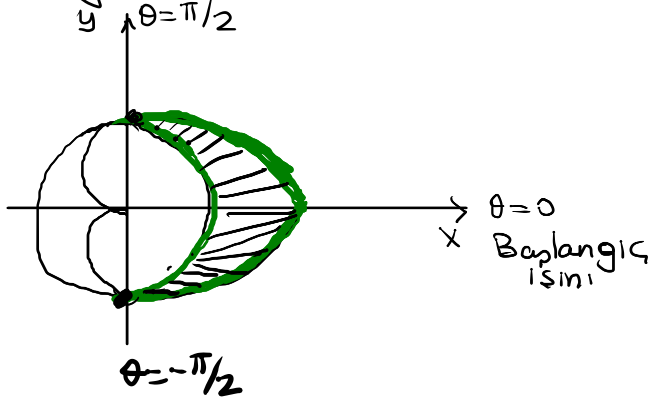
$2x=2y$

$x=y$

$$A=\iint_D dA=\iint_D r\,dr\,d\theta=\int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta+\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta=\frac{\pi-2}{2}\ \text{br}^2$$

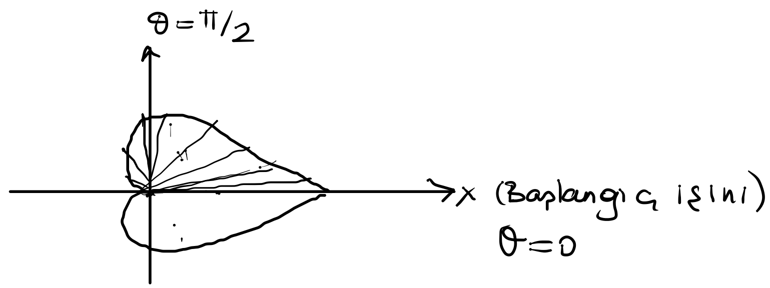
Ör/ $r=1+\cos\theta$ kardioidinin içinde $r=1$ çemberinin dışında kalan bölge üzerinde $f(x,y)$ nin integralini almak için integralin sınırlarını bulunuz.

$x=r\cos\theta$
$y=r\sin\theta$
$J=r$



$$I=\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ör/ $r=1+\cos\theta$ kardioidinin alanını hesaplayınız



$$\frac{A}{2}=\int_{0}^{\pi}\int_{0}^{1+\cos\theta} r\, dr\, d\theta$$

$$A=\frac{1}{2}\int_{0}^{2\pi} \rho^{2}\, d\rho\, d\theta$$

İki katlı integraller için ortalama değer teoremi

Düzlemde kapalı bir bölge üzerinde tanımlanmış iki değişkenli integrallenebilir bir fonksiyon için ortalama değer;

$$\hat{f}(x,y) = \frac{\iint_D f(x,y)\,dA}{\iint_D dA}$$

dır.

$$\hat{f}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

kapalı aralıkta verilmiş tek değişkenli integre edilebilir bir fonksiyonun ortalama değeri

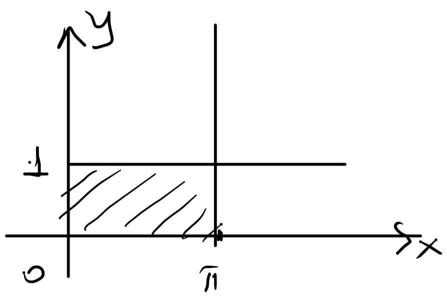
Ör/ $f(x,y) = x\cos(xy)$ 'nin $0 \le x \le \pi$, $0 \le y \le 1$ dikdörtgeni üzerinde ortalama değerini bulunuz.

$$\hat{f}(x,y) = \frac{\iint_D f(x,y)\,dA}{\iint_D dA} = \frac{\int_0^{\pi}\int_0^{1} x\cos(xy)\,dy\,dx}{\int_0^{\pi}\int_0^{1} dy\,dx} = \frac{\int_0^{\pi}\left[\frac{\cancel{x}}{\cancel{x}}\sin(xy)\Big|_0^1\right]dx}{\int_0^{\pi}\left(y\Big|_0^1\right)dx} = \frac{\int_0^{\pi}\sin x\,dx}{\int_0^{\pi}dx}$$

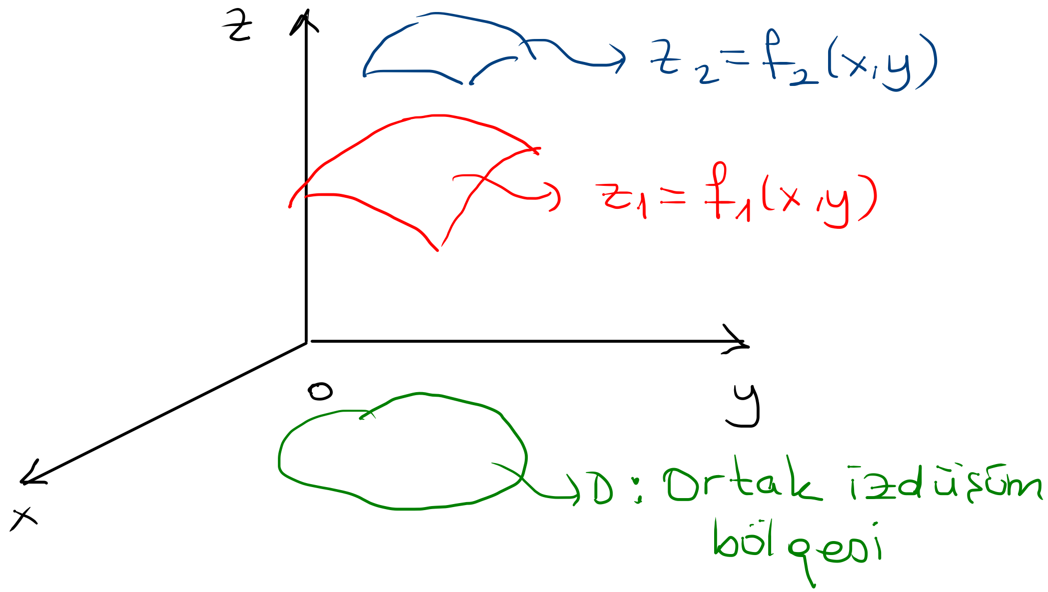
$$= \frac{-\cos x\Big|_0^{\pi}}{x\Big|_0^{\pi}}$$

$$= \frac{-(\cos\pi - \cos 0)}{(\pi - 0)}$$

$$= \frac{-(-1-1)}{\pi} = \frac{2}{\pi}$$

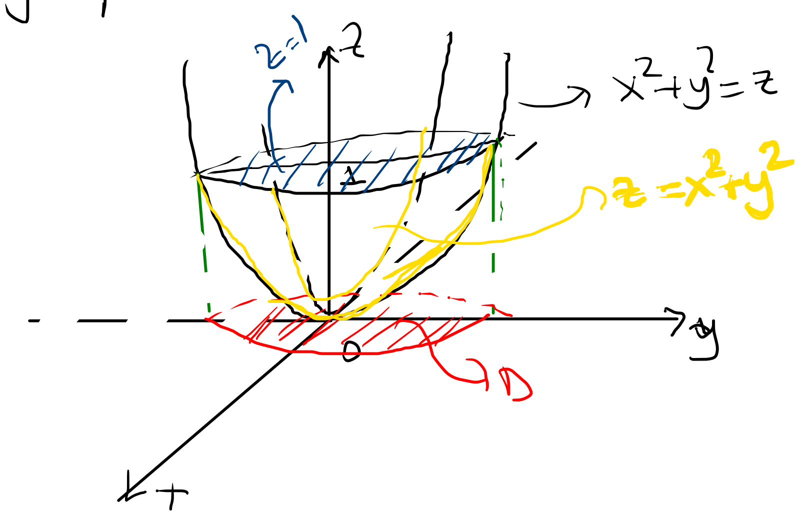


İki Yüzey Arasında Kalan Cismin Hacmi

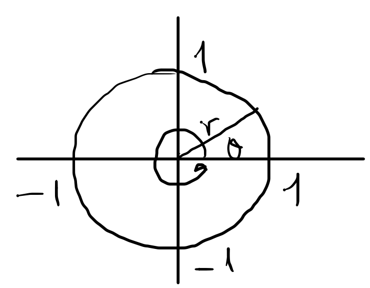


$$V = \iint_D (z_2 - z_1)\, dA$$

Ör: $z = x^2 + y^2$ paraboloidinin $z = 1$ ile kesilmiş kısmının hacmini hesaplayınız.



$x^2 + y^2 = 1$: D



$x = r\cos\theta$
$y = r\sin\theta$
$J = r$

$$V = \iint_D \left[1 - (x^2 + y^2)\right] dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \frac{\pi}{2}\ br^3$$