Bileşik Fonksiyonlar ve Türevleri

Tanım: $z=f(x,y)$ fonksiyonundaki x ve y değişkenleri başka bir değişkenin ya da değişkenlerin fonksiyonları iseler o zaman $z=f(x,y)$ fonksiyonuna bileşik fonksiyon denir.

Örneğin;

- $z=f(x,y)$ fonksiyonunda $x=h(t)$, $y=g(t)$ ise;

  $z=f[h(t),g(t)]=F(t)$ şeklinde t'ye bağlı bir bileşik fonksiyondur.

- $z=f(x,y)$ $\quad x=h(t,s)$ $\quad y=g(t,s)$

  $z=f[h(t,s),g(t,s)]=F(t,s)$

- $z=f(x,y)$ $\quad x=h(t)$, $y=g(r,s,t)$

  $z=f[h(t),g(r,s,t)]=F(r,s,t)$

$z = \arctan \frac{y}{x} \rightarrow z = f(x,y)$

$$\left.\begin{array}{ll} x = e^t \rightarrow & x = h(t) \\ y = \ln(t+rs) \rightarrow & y = g(r,s,t) \end{array}\right\} \Rightarrow z = F(r,s,t)$$

Teorem: $z = f(x,y)$, $x = x(t,s)$, $y = y(t,s)$ fonksiyonları ve bunların birinci mertebeden kısmi türevleri bir (a,b) noktasında tanımlı ve sürekli iseler o zaman

$$z = f(x,y) = f[x(t,s), y(t,s)] = F(t,s)$$

$$\frac{\partial z(a,b)}{\partial t} = \frac{\partial z(a,b)}{\partial x} \cdot \frac{\partial x(a,b)}{\partial t} + \frac{\partial z(a,b)}{\partial y} \cdot \frac{\partial y(a,b)}{\partial t}$$

$$\frac{\partial z(a,b)}{\partial s} = \frac{\partial z(a,b)}{\partial x} \cdot \frac{\partial x(a,b)}{\partial s} + \frac{\partial z(a,b)}{\partial y} \cdot \frac{\partial y(a,b)}{\partial s}$$

İspat:

$$\frac{\partial z(a,b)}{\partial t} = \lim_{\Delta t\to 0} \frac{\Delta z}{\Delta t}$$

$$f'(c) = \frac{f(b)-f(a)}{\underbrace{b-a}_{\Delta x}}$$

Δz artımını hesaplayalım.

$$\Delta x = x(a+\Delta t, b) - x(a,b) \qquad \Delta y = y(a+\Delta t, b) - y(a,b) \qquad \lim_{\Delta t\to 0} \Delta x = 0 \ ; \ \lim_{\Delta t\to 0} \Delta y = 0$$

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$\Delta z = f'_x(a+\theta_1\Delta x, b) \cdot \Delta x + f'_y(a+\Delta x, b+\theta_2\Delta y)\Delta y \qquad (\text{O.D.T})$$

$$\frac{\Delta z}{\Delta t} = f'_x(a+\theta_1\Delta x, b) \cdot \frac{\Delta x}{\Delta t} + f'_y(a+\Delta x, b+\theta_2\Delta y) \cdot \frac{\Delta y}{\Delta t}$$

$$\lim_{\Delta t\to 0} \frac{\Delta z}{\Delta t} = \left[\lim_{\Delta t\to 0} f'_x(a+\theta_1\Delta x, b)\right]\left(\lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t}\right) + \left[\lim_{\Delta t\to 0} f'_y(a+\Delta x, b+\theta_2\Delta y)\right]\left(\lim_{\Delta t\to 0} \frac{\Delta y}{\Delta t}\right)$$

$$\lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t\to 0} \frac{x(a+\Delta t, b) - x(a,b)}{\Delta t} = \frac{\partial x(a,b)}{\partial t}, \quad \lim_{\Delta t\to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t\to 0} \frac{y(a+\Delta t, b) - y(a,b)}{\Delta t} = \frac{\partial y(a,b)}{\partial t}$$

$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\left[\lim_{\Delta t\to 0} f'_x(a+\theta_1\Delta x, b)\right]\left(\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\right)+\left[\lim_{\Delta t\to 0} f'_y(a+\Delta x, b+\theta_2\Delta y)\right]\left(\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\right)$$

$$= f'_x(a,b)\cdot\frac{\partial x(a,b)}{\partial t}+f'_y(a,b)\cdot\frac{\partial y(a,b)}{\partial t}$$

$$=\frac{\partial z(a,b)}{\partial x}\,\frac{\partial x(a,b)}{\partial t}+\frac{\partial z(a,b)}{\partial y}\cdot\frac{\partial y(a,b)}{\partial t}$$

- $u=f(x,y,z) \qquad x=x(t)\ ,\ y=y(t,s)\ ,\ z=z(t,s,r)$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot 0+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot 0+\frac{\partial u}{\partial y}\cdot 0+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$z = f(x,y) \qquad \left.\begin{aligned} x &= x(t,s) \\ y &= y(t,s) \\ t &= t(r) \\ s &= s(r) \end{aligned}\right\} \Rightarrow z = f\big(x(t,s), y(t,s)\big)$$

$$= F(t,s)$$

$$= F(t(r), s(r))$$

$$= F_1(r)$$

$$x(t,s) = x(t(r), s(r)) = X(r)$$

$$y(t,s) = y(t(r), s(r)) = Y(r)$$

$$\frac{dz}{dr} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dr}$$

$$\left.\begin{aligned} \frac{dx}{dr} &= \frac{\partial x}{\partial t} \cdot \frac{dt}{dr} + \frac{\partial x}{\partial s} \cdot \frac{ds}{dr} \\ \frac{dy}{dr} &= \frac{\partial y}{\partial t} \cdot \frac{dt}{dr} + \frac{\partial y}{\partial s} \cdot \frac{ds}{dr} \end{aligned}\right\} \Rightarrow \frac{dz}{dr} = \frac{\partial z}{\partial x}\left[\frac{\partial x}{\partial t} \cdot \frac{dt}{dr} + \frac{\partial x}{\partial s} \cdot \frac{ds}{dr}\right] + \frac{\partial z}{\partial y}\left[\frac{\partial y}{\partial t} \cdot \frac{dt}{dr} + \frac{\partial y}{\partial s} \cdot \frac{ds}{dr}\right]$$

Ör/

$$\left.\begin{aligned} u &= xy+z \\ x &= t^2+2s \\ y &= xt \\ z &= 2y+s \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial t} = ?$$

$$\left.\begin{aligned} &x = x(t,s) \\ &y = y(x,t) = y(x(t,s),t) = Y(t,s) \\ &z = z(y,s) = z(Y(t,s),s) = Z(t,s) \end{aligned}\right\} u = f(x,y,z) = f(x(t,s), Y(t,s), Z(t,s)) = F(t,s)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\boxed{\frac{\partial y}{\partial t}} + \frac{\partial u}{\partial z}\cdot\boxed{\frac{\partial z}{\partial t}}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\cdot 1 \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial s}\cdot 0$$

$$= \frac{\partial z}{\partial y}\cdot\left[\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\right]$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\left[\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\right] + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\left[\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\right] = y\, 2t + x\cdot(t\cdot 2t + x) + 1\cdot 2\cdot[t\cdot 2t + x]$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\left[\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\right]+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\left[\frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\right]=y\,2t+x\cdot(t\cdot 2t+x)+1\cdot 2\cdot[t\cdot 2t+x]$$

$$=2yt+2t^2x+x^2+4t^2+2x$$

$$=2xt^2+2t^2x+x^2+4t^2+2x$$

$$=4(t^2+2s)\cdot t^2+(t^2+2s)^2+4t^2+2(t^2+2s)$$

Or/

$$\left.\begin{aligned}z&=\ln(x^2+y^2)\\ x&=e^u\cos v\\ y&=e^u\sin v\end{aligned}\right\}\Rightarrow \frac{\partial z}{\partial u}=?\qquad \frac{\partial z}{\partial v}=?$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$=\frac{2x}{x^2+y^2}\cdot(e^u\cos v)+\frac{2y}{x^2+y^2}\cdot(e^u\sin v)$$

$$=\frac{2e^{2u}\cos^2 v}{e^{2u}\cos^2 v+e^{2u}\sin^2 v}+\frac{2e^{2u}\sin^2 v}{e^{2u}\cos^2 v+e^{2u}\sin^2 v}$$

$$=2$$

$$\left.\begin{aligned}z&=f(x,y)\\ x&=x(u,v)\\ y&=y(u,v)\end{aligned}\right\}\Rightarrow z=f\left[x(u,v),y(u,v)\right]=F(u,v)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{2x}{x^2+y^2}\cdot(-e^u\sin v)+\frac{2y}{x^2+y^2}\cdot(e^u\cos v)$$

$$=\frac{2e^u\cos v\cdot(-e^u\sin v)+2(e^u\sin v)\cdot(e^u\cos v)}{e^{2u}\cos^2 v+e^{2u}\sin^2 v}=0$$

d/ $w = x + 2y + z^2$

$x = \frac{r}{s}$

$y = r^2 + \ln s$

$z = 2r$

$\frac{\partial w}{\partial s} = ?$

$\frac{\partial w}{\partial r} = ?$

$w = w(x, y, z)$

$x = x(r, s)$

$y = y(r, s)$

$z = z(r)$

$w = w\left[x(r,s), y(r,s), z(r)\right] = W(r, s)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot 0$$

$$= 1 \cdot \left(\frac{-r}{s^2}\right) + 2 \cdot \frac{1}{s}$$

$$= \frac{-r}{s^2} + \frac{2}{s}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dr}$$

$$= 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$= \frac{1}{s} + 4r + 4 \cdot (2r)$$

$$= \frac{1}{s} + 12r$$