- $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ denkleminde $x=r\cos\theta$, $y=r\sin\theta$ değişken dönüşümünü yaparak denklemin yeni şeklini bulunuz.

$$x=r\cos\theta$$
$$y=r\sin\theta$$
$$r=\sqrt{x^2+y^2}$$
$$\theta=\arctan\frac{y}{x}$$

$$\left.\begin{array}{l} z=f(x,y)\\ x=x(r,\theta)\\ y=y(r,\theta)\end{array}\right\} \Rightarrow z=F(r,\theta)$$

$$\left.\begin{array}{l} z=f(r,\theta)\\ r=r(x,y)\\ \theta=\theta(x,y)\end{array}\right\} z=F(x,y)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial z}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial z}{\partial \theta}\cdot\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{\partial z}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}}-\frac{\partial z}{\partial \theta}\cdot\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2}=\left[\frac{\partial^2 z}{\partial r^2}\cdot\frac{\partial r}{\partial x}+\frac{\partial^2 z}{\partial\theta\partial r}\cdot\frac{\partial\theta}{\partial x}\right]\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\sqrt{x^2+y^2}-\frac{x}{\sqrt{x^2+y^2}}\cdot x}{x^2+y^2}\cdot\frac{\partial z}{\partial r}-\left[\frac{\partial^2 z}{\partial r\partial\theta}\cdot\frac{\partial r}{\partial x}+\frac{\partial^2 z}{\partial\theta^2}\cdot\frac{\partial\theta}{\partial x}\right]\cdot\frac{y}{x^2+y^2}$$

$$+\frac{2xy}{(x^2+y^2)^2}\cdot\frac{\partial z}{\partial\theta}$$

$$=\frac{x^2}{x^2+y^2}\cdot\frac{\partial^2 z}{\partial r^2}-\frac{xy}{(x^2+y^2)\cdot\sqrt{x^2+y^2}}\frac{\partial^2 z}{\partial\theta\partial r}+\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\frac{\partial z}{\partial r}-\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\cdot\frac{\partial^2 z}{\partial r\partial\theta}+\frac{y^2}{(x^2+y^2)^2}\frac{\partial^2 z}{\partial\theta^2}$$

$$+\frac{2xy}{(x^2+y^2)^2}\frac{\partial z}{\partial\theta}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial r} = \frac{\partial z}{\partial r}\cdot\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\cdot\frac{1/x}{1+\frac{y^2}{x^2}} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial z}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{x^2+y^2} - y\cdot\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2}\cdot\frac{\partial z}{\partial r} + \frac{y}{\sqrt{x^2+y^2}}\left[\frac{\partial^2 z}{\partial r^2}\cdot\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial\theta\partial r}\cdot\frac{\partial\theta}{\partial y}\right] + \frac{-2xy}{(x^2+y^2)^2}\cdot\frac{\partial z}{\partial\theta}$$

$$+ \frac{x}{x^2+y^2}\left[\frac{\partial^2 z}{\partial r\partial\theta}\cdot\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial\theta^2}\cdot\frac{\partial\theta}{\partial y}\right]$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}\cdot\frac{\partial z}{\partial r} + \frac{y^2}{x^2+y^2}\frac{\partial^2 z}{\partial r^2} + \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\frac{\partial^2 z}{\partial\theta\partial r}} - \frac{2xy}{(x^2+y^2)^2}\frac{\partial z}{\partial\theta}$$

$$+ \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\frac{\partial^2 z}{\partial r\partial\theta}} + \frac{x^2}{(x^2+y^2)^2}\frac{\partial^2 z}{\partial\theta^2}$$

$$\frac{x^2}{x^2+y^2}\cdot\frac{\partial^2 z}{\partial r^2} - \cancel{\frac{xy}{(x^2+y^2)\cdot\sqrt{x^2+y^2}}\frac{\partial^2 z}{\partial\theta\partial r}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\frac{\partial z}{\partial r} - \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\cdot\frac{\partial^2 z}{\partial r\partial\theta}} + \frac{y^2}{(x^2+y^2)^2}\frac{\partial^2 z}{\partial\theta^2}$$

$$+ \frac{2xy}{(x^2+y^2)^2}\frac{\partial z}{\partial\theta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} \cdot \frac{\partial z}{\partial r} + \frac{y^2}{x^2+y^2} \frac{\partial^2 z}{\partial r^2} + \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} \frac{\partial^2 z}{\partial \theta \partial r}} - \cancel{\frac{2xy}{(x^2+y^2)^2} \frac{\partial z}{\partial \theta}}$$

$$+ \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} \frac{\partial^2 z}{\partial r \partial \theta}} + \frac{x^2}{(x^2+y^2)^2} \frac{\partial^2 z}{\partial \theta^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^2}{x^2+y^2} \cdot \frac{\partial^2 z}{\partial r^2} - \cancel{\frac{xy}{(x^2+y^2)\cdot\sqrt{x^2+y^2}} \frac{\partial^2 z}{\partial \theta \partial r}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} \frac{\partial z}{\partial r} - \cancel{\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} \cdot \frac{\partial^2 z}{\partial r \partial \theta}} + \frac{y^2}{(x^2+y^2)^2} \frac{\partial^2 z}{\partial \theta^2}$$

$$+ \cancel{\frac{2xy}{(x^2+y^2)^2} \frac{\partial z}{\partial \theta}}$$

$+$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \frac{\cancel{x^2+y^2}}{\cancel{x^2+y^2}} \frac{\partial^2 z}{\partial r^2} + \frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}} \frac{\partial^2 z}{\partial \theta^2} + \frac{\cancel{x^2+y^2}}{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}} \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{x^2+y^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{\sqrt{x^2+y^2}} \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = 0$$

Çok değişkenli fonksiyonlarda Taylor ve Mclaurin Formülleri

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots + n a_n (x-x_0)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 2\cdot 3\cdot a_3(x-x_0) + \cdots + n(n-1)a_n(x-x_0)^{n-2} + \cdots$$

$$\vdots$$

$$f^{(n)}(x) = n(n-1)(n-2)\cdots 1\cdot a_n + \cdots$$

$$f(x_0) = a_0 \qquad f'(x_0) = a_1 \qquad f''(x_0) = 2a_2, \cdots, f^{(n)}(x_0) = n!\, a_n$$

$$a_0 = f(x_0) \qquad a_1 = f'(x_0) \qquad a_2 = \frac{f''(x_0)}{2!} \qquad a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots \quad \text{Taylor}$$

$$x_0 = 0 \Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \quad \text{Mclaurin}$$

Teorem: $z=f(x,y)$ fonksiyonu ve $(n+1)$. mertebeye kadar kısmi türevleri $h>0,\ k>0$ birer sayı olmak üzere bir (a,b) noktasının $|x-a|\leq h$, $|y-b|\leq k$ civarında tanımlı ve sürekli iseler

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(2)}f(a,b)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n)}f(a,b)+R_n$$

formülüne iki değişkenli fonksiyonlar için Taylor formülü denir. R_n kalan terimdir.

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n+1)}\cdot f(a+\theta h,b+\theta k)\quad (0<\theta<1)$$

İSPAT: $z=f(x,y)$ fonksiyonunda $x=a+ht$, $y=b+kt$ değişken dönüşümünü yapalım.

$$\left.\begin{aligned} z&=f(x,y)\\ x&=x(t)\\ y&=y(t)\end{aligned}\right\}\quad z=f(a+ht,b+kt)=F(t)$$

Elde edilen $F(t)$ fonksiyonu $t=0$ noktasında tanımlı, sürekli ve her mertebeden türeve sahiptir. O halde $F(t)$ için Mclaurin formülünü yazabiliriz.

$$F(t)=F(0)+F'(0)\,t+\frac{F''(0)}{2!}t^2+\cdots+\frac{F^{(n)}(0)}{n!}t^n+R_n \qquad R_n=\frac{t^{n+1}}{(n+1)!}F^{(n+1)}(\theta t) \qquad (0<\theta<1)$$

$t=1$ alınırsa;

$$F(1)=F(0)+F'(0)+\frac{F''(0)}{2!}+\cdots+\frac{F^{(n)}(0)}{n!}+\frac{1}{(n+1)!}F^{(n+1)}(\theta)$$

$$z=f(a+ht,b+kt)=F(t)$$

$$F(1)=f(a+h,b+k)$$

$$F(0)=f(a,b)$$

$$F'(t)=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=\frac{\partial f}{\partial x}\cdot h+\frac{\partial f}{\partial y}\cdot k$$

$$=\left(h\cdot\frac{\partial}{\partial x}+k\cdot\frac{\partial}{\partial y}\right)f(a+ht,b+kt)$$

$$F'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)$$

$$F''(t)=\left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{dx}{dt}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dy}{dt}\right]\cdot h+\left[\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{dx}{dt}+\frac{\partial^2 f}{\partial y^2}\cdot\frac{dy}{dt}\right]\cdot k$$

$$=h^2\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial y\partial x}+hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(2)}f(a+ht,b+kt)$$

$$F''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(2)}f(a,b)$$

$$\vdots$$

$$F^{(n)}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n)}f(a,b)$$

$$F^{(n+1)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n+1)}f(a+ht,b+kt)\longrightarrow t=\theta \text{ alınırsa} \Rightarrow F^{(n+1)}(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n+1)}f(a+\theta h,b+\theta k)$$

$$\Rightarrow \underbrace{f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(2)} f(a,b) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(n)} f(a,b) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(n+1)} f(a+\theta h, b+\theta k)}_{\text{Taylor Formülü}}$$

$a+h=x \Rightarrow h=x-a$

$b+k=y \Rightarrow k=y-b$

$$\Rightarrow \underbrace{f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{(2)} f(a,b) + \cdots + \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{(n)} f(a,b) + R_n}_{\text{Taylor Formülünün diğer yorumu}}$$

$$\underbrace{f(x,y) = f(0,0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^{(2)} f(0,0) + \cdots + \frac{1}{n!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^{(n)} f(0,0) + R_n}_{\text{Mclaurin Formülü}}$$

Ör/ $f(x,y)=\ln(3+xy^2)$ fonksiyonunu $(-2,1)$ noktası civarında Taylor formülü ile üçüncü mertebe türevler dahil olmak üzere yazınız.

$$f(x,y)=f(-2,1)+\left[(x+2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]f(-2,1)+\frac{1}{2!}\left[(x+2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]^{(2)}f(-2,1)$$

$$+\frac{1}{3!}\left[(x+2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]^{(3)}f(-2,1)+R_3$$

$$f(x,y)=\ln(3+xy^2)$$

$$\frac{\partial f}{\partial x}=\frac{y^2}{3+xy^2}\qquad \frac{\partial^2 f}{\partial x^2}=\frac{-y^4}{(3+xy^2)^2}\qquad \frac{\partial^2 f}{\partial y\partial x}=\frac{2y(3+xy^2)-2xy\cdot y^2}{(3+xy^2)^2}=\frac{6y}{(3+xy^2)^2}$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=\frac{-6y\cdot 2(3+xy^2)\cdot y^2}{(3+xy^2)^4}=\frac{-12y^3}{(3+xy^2)^3}$$

$$\frac{\partial f}{\partial y}=\frac{2xy}{3+xy^2}\qquad \frac{\partial^2 f}{\partial y^2}=\frac{2x(3+xy^2)-4x^2y^2}{(3+xy^2)^2}=\frac{6x-2x^2y^2}{(3+xy^2)^2}$$

$$\frac{\partial^3 f}{\partial x^3}=\frac{2(3+xy^2)\cdot y^2\cdot y^4}{(3+xy^2)^4}=\frac{2y^6}{(3+xy^2)^3}$$

$$\frac{\partial^3 f}{\partial y^3}=\frac{-4x^2y(3+xy^2)^2-2(3+xy^2)\cdot 2xy\cdot(6x-2x^2y^2)}{(3+xy^2)^4}$$

$$=\frac{-4x^2y(3+xy^2)-4xy(6x-2x^2y^2)}{(3+xy^2)^3}=\frac{-36x^2y+4x^3y^3}{(3+xy^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2y(3+xy^2) - 2xy \cdot y^2}{(3+xy^2)^2} = \frac{6y}{(3+xy^2)^2} \Rightarrow \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{6(3+xy^2)^2 - 2(3+xy^2) \cdot 2xy \cdot 6y}{(3+xy^2)^4}$$

$$= \frac{6(3+xy^2) - 24xy^2}{(3+xy^2)^3} = \frac{18 - 18xy^2}{(3+xy^2)^3}$$

$f(-2,1) = 0$

$f'_x(-2,1) = 1$, $f'_y(-2,1) = -4$, $f''_{xx} = -1$, $f''_{xy} = 6$, $f''_{yy} = -20$, $f'''_{xxx} = 2$, $f'''_{xyx} = -12$

$f'''_{xyy} = 54$ $\quad f'''_{yyy} = -176$

$$f(-2,1) = 0 + \left[(x+2) \cdot 1 + (y-1) \cdot (-4)\right] + \frac{1}{2!}\left[(x+2)^2 \cdot (-1) + 2(x+2)(y-1) \cdot 6 + (y-1)^2 \cdot (-20)\right]$$

$$+ \frac{1}{3!}\left[(x+2)^3 \cdot 2 + 3(x+2)^2(y-1) \cdot (-12) + 3(x+2)(y-1)^2 \cdot 54 + (y-1)^3 \cdot (-176)\right] + R_3$$

Taylor Formülünün Önemli Bir Sonucu

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(2)}f(a,b)+\dots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(n)}f(a,b)+R_n$$

$n=0$ düşünülürse

$$f(a+h,b+k)=f(a,b)+R_0$$

$$R_0=\frac{1}{(0+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{(0+1)}f(a+\theta h,b+\theta k)=h\cdot\frac{\partial f(a+\theta h,b+\theta k)}{\partial x}+k\cdot\frac{\partial f(a+\theta h,b+\theta k)}{\partial y}$$

$$f(a+h,b+k)-f(a,b)=h\cdot\frac{\partial f(a+\theta h,b+\theta k)}{\partial x}+k\cdot\frac{\partial f(a+\theta h,b+\theta k)}{\partial y}\quad(*)$$

olur. Elde edilen bu eşitlik ortalama değer teoremine benzemekle beraber daha iyi bir sonuçtur. $h=\Delta x$, $k=\Delta y$ alınırsa, ortalama değer teoremindeki θ_1 ve θ_2 gibi sıfır ile bir arasındaki iki sayıya karşılık tek bir θ sayısı bu formülde bulunur.

Bu sonucun uygulaması

$f(x,y)$ ve $g(x,y)$ fonksiyonları ve bunların birinci mertebeden türevleri tanımlı ve sürekli olsun. $f(0,0)=0$, $g(0,0)=0$ olmak üzere

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{f(x,y)}{g(x,y)}$ limitine $y=\lambda x$ doğrusu boyunca yaklaşmak isteyelim.

(*) denkleminde $a=0$, $b=0$, $h=x$, $k=y$ alalım.

$$f(x,y)-\underbrace{f(0,0)}_{=0} = x\cdot f'_x(\theta_1 x,\theta_1 y)+y\cdot f'_y(\theta_1 x,\theta_1 y) \Rightarrow f(x,y)=x\cdot f'_x(\theta_1 x,\theta_1 y)+y\, f'_y(\theta_1 x,\theta_1 y)$$

$$g(x,y)-\underbrace{g(0,0)}_{=0} = x\cdot g'_x(\theta_2 x,\theta_2 y)+y\, g'_y(\theta_2 x,\theta_2 y) \Rightarrow g(x,y)=x\cdot g'_x(\theta_2 x,\theta_2 y)+y\, g'_y(\theta_2 x,\theta_2 y)$$

$$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y)}{g(x,y)} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x\cdot f'_x(\theta_1 x,\theta_1 y)+y\, f'_y(\theta_1 x,\theta_1 y)}{x\, g'_x(\theta_2 x,\theta_2 y)+y\, g'_y(\theta_2 x,\theta_2 y)} = \lim\limits_{\substack{x\to 0\\ y=\lambda x}} \frac{x\cdot f'_x(\theta_1 x,\theta_1\lambda x)+\lambda x\cdot f'_y(\theta_1 x,\lambda\theta_1 x)}{x\cdot g'_x(\theta_2 x,\lambda\theta_2 x)+\lambda x\, g'_y(\theta_2 x,\lambda\theta_2 x)}$$

$$= \lim\limits_{x\to 0} \frac{f'_x(\theta_1 x,\lambda\theta_1 x)+\lambda f'_y(\theta_1 x,\lambda\theta_1 x)}{g'_x(\theta_2 x,\lambda\theta_2 x)+\lambda g'_y(\theta_2 x,\lambda\theta_2 x)}$$

$$= \frac{f'_x(0,0)+\lambda f'_y(0,0)}{g'_x(0,0)+\lambda g'_y(0,0)} \qquad \left(g'_x(0,0)+\lambda g'_y(0,0)\neq 0\right)$$

Bu limitin λ'dan bağımsız olabilmesi için λ'ya göre türevi sıfır olmalıdır. (Yoldan bağımsız olması için)

Bu sonuçtan λ'ya göre türev alıp sıfıra eşitlendiğinde $\left.\frac{D(f,g)}{D(x,y)}\right|_{(0,0)} = 0$ olduğu elde edilir.

Ör/ $A = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{\sin(xy) + xe^x - y}{x\cos y + \sin 2y}$ limitini $y = -x$ doğrusu boyunca hesaplayınız.

$\lambda = -1$

$f(x,y) = \sin(xy) + xe^x - y \qquad f'_x = y\cos(xy) + e^x + xe^x \qquad f'_y = x\cos(xy) - 1$

$g(x,y) = x\cos y + \sin 2y \qquad g'_x = \cos y \qquad g'_y = -x\sin y + 2\cos 2y$

$f'_x(0,0) = 1 \qquad f'_y = -1$

$g'_x(0,0) = 1 \qquad g'_y = 2$

$A = \dfrac{1-(-1)}{1-2} = \dfrac{2}{-1} = -2$

$\left.\dfrac{D(f,g)}{D(x,y)}\right|_{(0,0)} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 + 1 = 3 \neq 0$ olduğundan limit λ değerine bağlıdır. Yani yola bağlıdır.