Ör/ Sonlu $[a,b]$ aralığında verilmiş

$$-y''(x) = \lambda y(x) \qquad a \le x \le b \qquad (1)$$
$$y(a) = y(b) = 0$$

Sınır değer probleminin özdeğer ve özfonksiyonlarını bulalım.

$$-y'' + q(x)y = \lambda y \qquad q(x) \equiv 0, \quad \alpha = 0, \quad \beta = 0$$
$$y(a)\cos\alpha + y'(a)\sin\alpha = 0$$
$$y(b)\cos\beta + y'(b)\sin\beta = 0$$

$$-y'' = \lambda y \Rightarrow y'' + \lambda y = 0 \qquad \text{K.D}: r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda$$

2°) $\lambda = 0 \Rightarrow r^2 = 0 \Rightarrow r_{1,2} = 0$

$y = A.e^0 + B.e^0.x = A + Bx$

$y(a) = 0 \Rightarrow A + Ba = 0 \Rightarrow A = -Ba$

$y(b) = 0 \Rightarrow A + Bb = 0 \Rightarrow Bb - Ba = 0$

$\underbrace{B(b-a)}_{\neq 0} = 0 \Rightarrow B = 0, \; A = 0$

Sadece aşikâr çözüm var.

3°) $\lambda < 0 \Rightarrow \lambda = -\mu^2 < 0 \Rightarrow r^2 - \mu^2 = 0 \Rightarrow r^2 = \mu^2$

$r = \mp\mu$

$y = c_1 e^{-\mu x} + c_2 e^{\mu x}$

$y(a) = c_1 e^{-\mu a} + c_2 e^{\mu a} = 0$

$y(b) = c_1 e^{-\mu b} + c_2 e^{\mu b} = 0$

$$\begin{vmatrix} e^{-\mu a} & e^{\mu a} \\ e^{-\mu b} & e^{\mu b} \end{vmatrix} = e^{\mu(b-a)} - e^{\mu(a-b)} \neq 0$$

Sadece aşikar çözüm vardır.

1°) $\lambda > 0 \Rightarrow r = \mp\sqrt{\lambda}\, i \Rightarrow y = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$

$$\left.\begin{aligned} y(a) = 0 &\Rightarrow c_1\cos\sqrt{\lambda}\,a + c_2\sin\sqrt{\lambda}\,a = 0 \\ y(b) = 0 &\Rightarrow c_1\cos\sqrt{\lambda}\,b + c_2\sin\sqrt{\lambda}\,b = 0 \end{aligned}\right\}$$

sistemin aşikar çözümden farklı çözümlerinin olması için katsayılar matrisinin determinantı sıfır olmalıdır.

$$\begin{vmatrix} \cos\sqrt{\lambda}\,a & \sin\sqrt{\lambda}\,a \\ \cos\sqrt{\lambda}\,b & \sin\sqrt{\lambda}\,b \end{vmatrix} = 0$$

$$\underbrace{\cos\sqrt{\lambda}a \cdot \sin\sqrt{\lambda}b - \cos\sqrt{\lambda}b \sin\sqrt{\lambda}a}_{\sin\sqrt{\lambda}(b-a)} = 0 \Rightarrow \sin\sqrt{\lambda}(b-a) = 0 \Rightarrow \sqrt{\lambda}(b-a) = n\pi \quad (n = 0, \mp 1, \mp 2, \dots)$$

$$\sqrt{\lambda} = \frac{n\pi}{b-a}$$

$$\boxed{\lambda = \frac{n^2\pi^2}{(b-a)^2}}$$

$$\left.\begin{aligned} c_1\cos\sqrt{\lambda}a + c_2\sin\sqrt{\lambda}a &= 0 \\ c_1\cos\sqrt{\lambda}b + c_2\sin\sqrt{\lambda}b &= 0 \end{aligned}\right\} \rightarrow c_1\cos\left(\frac{n\pi a}{b-a}\right) + c_2\sin\left(\frac{n\pi a}{b-a}\right) = 0$$

$$c_1 = -c_2\tan\left(\frac{n\pi a}{b-a}\right)$$

$$y = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$$

$$= -c_2\tan\left(\frac{n\pi a}{b-a}\right)\cdot\cos\left(\frac{n\pi x}{b-a}\right) + c_2\sin\left(\frac{n\pi x}{b-a}\right)$$

$$= \frac{c_2}{\cos\left(\frac{n\pi a}{b-a}\right)}\left[-\sin\left(\frac{n\pi a}{b-a}\right)\cdot\cos\left(\frac{n\pi x}{b-a}\right) + \sin\left(\frac{n\pi x}{b-a}\right)\cdot\cos\left(\frac{n\pi a}{b-a}\right)\right]$$

$$= \frac{c_2}{\cos\left(\frac{n\pi a}{b-a}\right)}\cdot\sin\frac{n\pi(x-a)}{b-a} = A\cdot\sin\left(\frac{n\pi(x-a)}{b-a}\right) \rightarrow$$ sistemin 1. denkleminden elde edilen sonuç.

Bir sabittir.

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2}, \quad y_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

- $$c_1 = -c_2\tan\left(\frac{n\pi b}{b-a}\right) \Rightarrow y = \frac{c_2}{\cos\left(\frac{n\pi b}{b-a}\right)}\cdot\sin\left(\frac{n\pi(x-b)}{b-a}\right) = B\cdot\sin\left(\frac{n\pi(x-b)}{b-a}\right)$$

slbt.

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2} \quad y_n = \sin\left(\frac{n\pi(x-b)}{b-a}\right)$$

$$u_n = \sin\left(\frac{n\pi(x-b)}{b-a}\right) = \sin\left(\frac{n\pi(x-b+a-a)}{b-a}\right)$$

$$= \sin\left[\frac{n\pi[(x-a)-(b-a)]}{b-a}\right]$$

$$= \sin\left[n\pi\left(\frac{x-a}{b-a}-1\right)\right]$$

$$= (-1)^n \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

$$= (-1)^n \cdot y_n$$

$$v_n(x) = \frac{y_n}{\|y_n\|} \quad , \quad \|y_n\|^2 = \int_a^b \sin^2\left(\frac{n\pi(x-a)}{b-a}\right) dx = \int_a^b \frac{1-\cos\left(\frac{2n\pi(x-a)}{b-a}\right)}{2}\, dx$$

$$= \frac{1}{2}\int_a^b dx - \frac{1}{2}\int_a^b \cos\left(\frac{2n\pi(x-a)}{b-a}\right) dx$$

$$= \frac{x}{2}\Big|_a^b - \underbrace{\left[\frac{1}{2}\cdot\frac{b-a}{2n\pi}\sin\left(\frac{2n\pi(x-a)}{b-a}\right)\Big|_a^b\right]}_{0}$$

$$= \frac{b-a}{2}$$

$$\|y_n\| = \sqrt{\frac{b-a}{2}} \Rightarrow \frac{1}{\|y_n\|} = \sqrt{\frac{2}{b-a}}$$

$$\boxed{v_n(x) = \frac{\sqrt{2}}{\sqrt{b-a}} \cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right)}$$