Ör/ $y''' + y' = \sec^2 x$ dif. denkleminin genel çözümünü bulunuz.

$(D^3 + D)\, y = \sec^2 x$

K.D: $r^3 + r = 0$

$r(r^2+1) = 0$

$r_1 = 0 \quad r_{2,3} = \mp i$

$y_h = c_1 + c_2 \cos x + c_3 \sin x$

$c_1 = c_1(x)$

$c_2 = c_2(x)$

$c_3 = c_3(x)$

$$\begin{cases} ① & c_1' + c_2' \cos x + c_3' \sin x = 0 \quad (\text{keyfi}) \\ ② & -c_2' \sin x + c_3' \cos x = 0 \quad (\text{keyfi}) \\ ③ & -c_2' \cos x - c_3' \sin x = \sec^2 x \end{cases}$$

① + ③ 'den $\quad c_1' = \sec^2 x \Rightarrow c_1 = \int \sec^2 x \, dx + k_1$

$$\boxed{c_1 = \tan x + k_1}$$

② . $\sin x$ + ③ . $\cos x$ 'den

$$-c_2' \sin^2 x - c_2' \cos^2 x = \frac{1}{\cos^2 x} \cdot \cos x$$

$$-c_2' = \frac{1}{\cos x} \Rightarrow c_2' = \frac{-1}{\cos x} \Rightarrow c_2 = \int -\sec x \, dx + k_2$$

$$\boxed{c_2 = -\ln|\sec x + \tan x| + k_2}$$

$(\cos x)\cdot 2 + (\sin x)\cdot 3$ 'den

$$\Rightarrow -c_3' \cos^2 x - c_3' \sin^2 x = \frac{\sin x}{\cos^2 x}$$

$$-c_3' = \frac{\sin x}{\cos^2 x} \Rightarrow c_3' = \frac{-\sin x}{\cos^2 x}$$

$\cos x = u$

$-\sin x \, dx = du$

$$\Rightarrow c_3 = -\int \frac{\sin x}{\cos^2 x}\, dx + k_3$$

$$= \int \frac{du}{u^2} + k_3$$

$$= -\frac{1}{u} + k_3 = -\frac{1}{\cos x} + k_3$$

$$\Rightarrow \boxed{c_3 = -\sec x + k_3}$$

$$y = c_1 + c_2 \cos x + c_3 \sin x = \tan x + k_1 + \left[-\ln|\sec x + \tan x| + k_2\right]\cos x + \left[-\sec x + k_3\right]\cdot \sin x$$

$$= \underbrace{k_1 + k_2 \cos x + k_3 \sin x}_{y_h} + \underbrace{\tan x - \cos x \cdot \ln|\sec x + \tan x| - \sec x \cdot \sin x}_{y_ö}$$

Ör/ $y'' - y = \dfrac{2}{e^x+1}$ dif. denklemin genel çözümünü bulunuz.

$$(D^2-1)y = \frac{2}{e^x+1}$$

K.D: $r^2-1=0$

$r_{1,2} = \mp 1$

$$y = c_1 e^{-x} + c_2 e^x$$

$c_1 = c_1(x)$

$c_2 = c_2(x)$

$$c_1' e^{-x} + c_2' e^x = 0 \quad (\text{keyfi})$$

$$-c_1' e^{-x} + c_2' e^x = \frac{2}{e^x+1}$$

$$2c_2' e^x = \frac{2}{e^x+1}$$

$$c_2' = \frac{e^{-x}}{e^x+1} \Rightarrow c_2 = \int \frac{e^{-x}}{e^x+1}\,dx$$

$e^x + 1 = u \Rightarrow e^x = u-1$

$e^x\,dx = du \qquad e^{-x} = \dfrac{1}{u-1}$

$dx = \dfrac{du}{u-1}$

$$= \int \frac{1}{u-1} \cdot \frac{1}{u} \cdot \frac{du}{u-1}$$

$$= \int \frac{du}{u(u-1)^2} = \int \left[\frac{1}{u} - \frac{1}{u-1} + \frac{1}{(u-1)^2}\right] du$$

$$\frac{1}{u(u-1)^2} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{(u-1)^2}$$

$$A(u-1)^2 + Bu(u-1) + Cu \equiv 1 \Rightarrow A(u^2-2u+1) + B(u^2-u) + Cu \equiv 1$$

$A + B = 0$

$-2A - B + C = 0$

$A = 1 \Rightarrow B = -1 \Rightarrow C = 1$

$$= \int \left[\frac{1}{u} - \frac{1}{u-1} + \frac{1}{(u-1)^2}\right] du$$

$$= \ln|u| - \ln|u-1| - \frac{1}{u-1} + k_2$$

$$\boxed{c_2 = \ln|e^x+1| - \underbrace{\ln e^x}_{x} - \frac{1}{e^x} + k_2}$$

$$e^x + 1 = u$$

$$e^x\,dx = du$$

$$c_1' e^{-x} + \left(\frac{e^{-x}}{e^x+1}\right) \cdot e^x = 0$$

$$c_1' e^{-x} = -\frac{1}{e^x+1}$$

$$c_1' = -\frac{e^x}{e^x+1}$$

$$c_1 = -\int \frac{e^x}{e^x+1}\,dx + k_1$$

$$= -\int \frac{du}{u} + k_1$$

$$= -\ln|u| + k_1$$

$$\boxed{c_1 = -\ln|e^x+1| + k_1}$$

$$y = c_1 e^{-x} + c_2 e^x$$

$$= -e^{-x}\ln|e^x+1| + k_1 e^{-x} + e^x \ln|e^x+1| - x e^x - 1 + k_2 e^x$$

$$= \underbrace{k_1 e^{-x} + k_2 e^x}_{y_h} + \underbrace{(e^x - e^{-x}) \cdot \ln|e^x+1| - x e^x - 1}_{y_ö}$$

Ödev:

$$y'' + 4y' + 5y = \frac{e^{-2x}}{\cos x} \qquad y_{G.Ç} = ?$$

Yüksek Mertebeden Değişken Katsayılı Diferansiyel Denklemler

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

şeklindeki denklemlerdir. Bunlardan özel olarak seçtiğimiz denklemleri çalışacağız.

Cauchy-Euler Dif. Denklemi

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = f(x)$$ şeklindeki denklemlerdir.

Bu dif. denklemi çözmek için $x=e^t$ dönüşümü yapılarak denklem sabit katsayılı dif. denkleme dönüştürülür.

$$\frac{d}{dt} = D$$

$$x = e^t \Rightarrow t = \ln x \qquad (x>0)$$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dt}{dt} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}} = \frac{dy}{dt}\cdot\frac{1}{e^t} = e^{-t}\cdot\frac{dy}{dt} = e^{-t}Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{d}{dt}\left[e^{-t}\frac{dy}{dt}\right]\cdot\frac{1}{\frac{dx}{dt}} = \left[-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right]\cdot\frac{1}{e^t}$$

$$= e^{-2t}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] = e^{-2t}\left[D^2y - Dy\right]$$

$$= e^{-2t}D(D-1)y$$

Ör/ $x^2y'' + 5xy' + 3y = 0$ Cauchy-Euler genel çöz. bulunuz.

$x = e^t$
$y' = e^{-t} Dy$
$y'' = e^{-2t} D(D-1)y$

$\Rightarrow \underbrace{e^{2t} \cdot [e^{-2t}}_{=1} D(D-1)y] + 5 \cdot \underbrace{e^t \cdot e^{-t}}_{=1} Dy + 3y = 0$

$[D(D-1) + 5D + 3]\, y = 0$

$(D^2 + 4D + 3)y = 0$

$x = e^t$
$\Rightarrow e^{-t} = \frac{1}{x}$
$\Rightarrow e^{-3t} = \frac{1}{x^3}$

K.D: $r^2 + 4r + 3 = 0$

$(r+1)(r+3) = 0$

$r_1 = -1$, $r_2 = -3$

$y = c_1 e^{-t} + c_2 e^{-3t}$

$$\boxed{y = \frac{c_1}{x} + \frac{c_2}{x^3}}$$

Ör/ $x^3 y''' + 2xy' - 2y = x^2 \ln x + 3x$ dif. denkleminin genel çözümünü bulunuz.

$$\left.\begin{aligned} & x = e^t \Rightarrow t = \ln x \\ & y' = e^{-t} D y \\ & y'' = e^{-2t} D(D-1) y \\ & y''' = e^{-3t} D(D-1)(D-2) y \end{aligned}\right\} \Rightarrow \underbrace{e^{3t}\left[e^{-3t}\right.}_{=1} D(D-1)(D-2)y\big] + 2.\underbrace{e^{t}.e^{-t}}_{=1} Dy - 2y = te^{2t} + 3e^t$$

$$\left[D(D-1)(D-2) + 2D - 2\right] y = te^{2t} + 3e^t$$

$$\left[D(D-1)(D-2) + 2(D-1)\right] y = \underbrace{te^{2t}}_{f_1} + \underbrace{3e^t}_{f_2} \longrightarrow \left[D^3 - 3D^2 + 2D + 2D - 2\right] y = te^{2t} + 3e^t$$

$$(D^3 - 3D^2 + 4D - 2) y = te^{2t} + 3e^t$$

K.D: $(r-1)\left[r(r-2) + 2\right] = 0$

$(r-1)\left[r^2 - 2r + 2\right] = 0$

$r_1 = 1 \qquad r_{2,3} = \dfrac{2 \mp \sqrt{4-8}}{2}$

$= 1 \mp i$

$$y_h = c_1 e^t + e^t . \left[c_2 \cos t + c_3 \sin t\right]$$

$$y_{ö1} = (at + b).e^{2t}$$

$$y'_{ö1} = a e^{2t} + 2(at+b) e^{2t}$$

$$y''_{ö} = 2ae^{2t} + 2ae^{2t} + 4(at+b)e^{2t} = 4ae^{2t} + 4(at+b)e^{2t}$$

$$y'''_{ö} = 8ae^{2t} + 4ae^{2t} + 8(at+b)e^{2t} = 12ae^{2t} + 8(at+b)e^{2t}$$

$$\Rightarrow 12ae^{2t} + 8(at+b)e^{2t} - 3\left[4ae^{2t} + 4(at+b)e^{2t}\right] + 4\left[ae^{2t} + 2(at+b)e^{2t}\right] - 2(at+b)e^{2t} \equiv te^{2t}$$

$$\left[4a + 2(at+b)\right] e^{2t} \equiv te^{2t} \Rightarrow 2at + (4a + 2b) \equiv t \Rightarrow 2a = 1 \ , \ 4a + 2b = 0$$

$2a = 1 \;,\; 4a + 2b = 0$

$\Rightarrow a = \frac{1}{2} \Rightarrow b = -1$

$$\boxed{y_{ö1} = \left(\frac{t}{2} - 1\right) e^{2t}}$$

$$\left.\begin{aligned} y_{ö2} &= Ate^t \\ y_{ö2}' &= Ae^t + Ate^t \\ y_{ö2}'' &= 2Ae^t + Ate^t \\ y_{ö2}''' &= 3Ae^t + Ate^t \end{aligned}\right\} \; 3Ae^t + Ate^t - 3(2Ae^t + Ate^t) + 4(Ae^t + Ate^t) - 2Ate^t \equiv 3e^t$$

$$Ae^t \equiv 3e^t \Rightarrow A = 3 \Rightarrow \boxed{y_{ö2} = 3te^t}$$

$$y_{G.Ç} = c_1 e^t + e^t\left[c_2 \cos t + c_3 \sin t\right] + \left(\frac{t}{2} - 1\right) e^{2t} + 3te^t$$

$$= c_1 x + x\left[c_2 \cos(\ln x) + c_3 \sin(\ln x)\right] + \left(\frac{\ln x}{2} - 1\right) \cdot x^2 + 3x \ln x$$

Ör/ $y'' - \frac{3}{x} y' + \frac{3}{x^2} y = \frac{1}{x^2}$ dif. denkleminin genel çözümünü bulunuz.

$x^2 y'' - 3xy' + 3y = 1$ Cauchy - Euler

$$\left.\begin{aligned} x &= e^t \\ y' &= e^{-t} Dy \\ y'' &= e^{-2t} D(D-1)y \end{aligned}\right\}$$

$$\underbrace{e^{2t} \cdot e^{-2t}}_{=1} D(D-1)y - 3 \underbrace{e^{t} \cdot e^{-t}}_{=1} Dy + 3y = 1$$

$$(D^2 - D - 3D + 3)\, y = 1$$

$$(D^2 - 4D + 3)\, y = 1$$

$$k.D = r^2 - 4r + 3 = 0$$

$$(r-3)(r-1) = 0$$

$$r_1 = 3 \quad r_2 = 1$$

$$y_h = c_1 e^{3t} + c_2 e^{t}$$

$$\left.\begin{aligned} y_ö &= a \\ y_ö' &= y_ö'' = 0 \end{aligned}\right\} \quad 3a = 1 \Rightarrow a = \frac{1}{3}$$

$$y_ö = \frac{1}{3}$$

$$y_{G.Ç} = y_h + y_ö = c_1 e^{3t} + c_2 e^{t} + \frac{1}{3}$$

$$= c_1 x^3 + c_2 x + \frac{1}{3}$$

$$A_0(ax+b)^n y^{(n)} + A_1(ax+b)^{n-1} \cdot y^{(n-1)} + \dots + A_{n-2}(ax+b)^2 y'' + A_{n-1}(ax+b) y' + A_n y = f(x)$$

$$\left(\frac{d}{dt} = D\right)$$

$$ax+b = e^t \Rightarrow x = \frac{e^t - b}{a}$$

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dt}{dt} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{1}{\frac{e^t}{a}} = a \cdot e^{-t} \cdot \frac{dy}{dt} = a e^{-t} D y$$

$$\frac{d^2y}{dx^2} = a^2 e^{-2t} D(D-1) y$$

$$\frac{d^3y}{dx^3} = a^3 e^{-3t} D(D-1)(D-2) y$$

$$\vdots$$

Ör/ $(x-1)^2 y'' - 2(x-1)y' - 4y = 0$ dif. denk'nin genel çözümünü bulunuz.

$$x-1 = e^t \Rightarrow x = e^t + 1$$

$$\left.\begin{aligned} y' &= e^{-t} D y \\ y'' &= e^{-2t} D(D-1) y \end{aligned}\right\} \underbrace{e^{2t} \cdot e^{-2t}}_{=1} D(D-1) y - 2 \underbrace{e^{t} \cdot e^{-t}}_{=1} D y - 4y = 0 \Rightarrow (D^2 - D - 2D - 4) y = 0$$

$$(D^2 - 3D - 4) y = 0$$

k.D: $r^2 - 3r - 4 = 0$

$-4 \quad 1$

$(r-4)(r+1) = 0$

$r_1 = 4 \quad r_2 = -1$

$y = c_1 e^{4t} + c_2 e^{-t}$

$$\boxed{y = c_1 \cdot (x-1)^4 + \frac{c_2}{x-1}}$$

$$\frac{d^3y}{dx^3} = e^{-3t} D(D-1)(D-2) y$$

$\vdots$

$$\frac{d^ny}{dx^n} = e^{-nt} D(D-1) \cdots (D-n+1) y$$

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = f(x)$$

$$a_0 \underbrace{e^{nt} \left[e^{-nt}\right.}_{1} D(D-1) \cdots (D-n+1) y \Big] + a_1 \cdot \underbrace{e^{(n-1)t} \cdot \left[e^{-(n-1)t}\right.}_{1} D(D-1) \cdots (D-n+2) y \Big] + \cdots + a_{n-1} \underbrace{e^{t} \cdot e^{-t}}_{1} D y + a_n y = f(e^t)$$

$$\left[a_0 D(D-1) \cdots (D-n+1) + a_1 D(D-1) \cdots (D-n+2) + \cdots + a_{n-1} D + a_n \right] y = f(e^t) \rightarrow$$

sabit katsayılı dif. denk.