— Uygulama —

1) $\int (ax^2+2bx)^4 (ax+b)\,dx = \int u^4 \frac{du}{2} = \frac{1}{2}\frac{u^5}{5} + c$

$ax^2+2bx = u$

$(2ax+2b)\,dx = du$

$(ax+b)\,dx = \frac{du}{2}$

$= \frac{1}{10}(ax^2+2bx)^5 + c$

2) $\int \frac{\sec^2(ax+b)\,dx}{\tan(ax+b)+c} = \int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{du}{u}$

$= \frac{1}{a}\ln|u| + k$

$\tan(ax+b)+c = u$

$a.\sec^2(ax+b)\,dx = du$

$\sec^2(ax+b)\,dx = \frac{du}{a}$

$= \frac{1}{a}\ln|\tan(ax+b)+c| + k$

3) $\int \sqrt{\sin x}\,\cos^3 x\,dx = \int \sqrt{\sin x}\,.\cos^2 x.\cos x\,dx$

$\sin x = u$

$\cos x\,dx = du$

$= \int \sqrt{\sin x}\,(1-\sin^2 x)\cos x\,dx$

$= \int \left[(\sin x)^{1/2} - (\sin x)^{5/2}\right].\cos x\,dx$

$= \int (u^{1/2} - u^{5/2})\,du$

$= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + c$

$= \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + c$

4) $\int \frac{(x+1)\,dx}{x^2-4x+8} = \frac{1}{2}\int \frac{2(x+1)}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{2x+2+4-4}{x^2-4x+8}\,dx$

$x^2-4x+8 = u$

$(2x-4)\,dx$

$= \frac{1}{2}\int \frac{2x-4}{x^2-4x+8}\,dx + \frac{6}{2}\int \frac{dx}{x^2-4x+8}$

$$= \frac{1}{2}\int \frac{du}{u} + 3\int \frac{dx}{(x-2)^2+4} \qquad (x-2)=t, \quad dx = dt$$

$$= \frac{1}{2}\ln|u| + 3\int \frac{dt}{t^2+4}$$

$$= \frac{1}{2}\ln|u| + \frac{3}{2}\arctan\frac{t}{2} + C$$

$$= \frac{1}{2}\ln|x^2-4x+8| + \frac{3}{2}\arctan\frac{(x-2)}{2} + C$$

5) $\displaystyle\int \frac{(x+3)\,dx}{\sqrt{5-4x-x^2}} = \int \frac{(x+3)\,dx}{\sqrt{-x^2-4x+5}} = -\frac{1}{2}\int \frac{-2(x+3)}{\sqrt{-x^2-4x+5}}\,dx$

$-x^2-4x+5 = u$

$(-2x-4)\,dx = du$

$$= -\frac{1}{2}\int \frac{(-2x-4)-2}{\sqrt{-x^2-4x+5}}\,dx$$

$$= -\frac{1}{2}\int \frac{(-2x-4)\,dx}{\sqrt{-x^2-4x+5}} + \int \frac{dx}{\sqrt{-(x^2+4x-5)}}$$

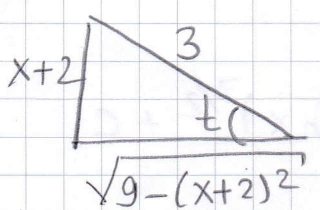
$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \int \frac{dx}{\sqrt{-[(x+2)^2-9]}}$$

$$= -\int \frac{du}{2\sqrt{u}} + \int \frac{dx}{\sqrt{9-(x+2)^2}}$$

$x+2 = 3\sin t$

$dx = 3\cos t\,dt$

$$= -\sqrt{u} + \int \frac{3\cos t\,dt}{\sqrt{9-9\sin^2 t}}$$



$$= -\sqrt{5-4x-x^2} + \int \frac{3\cos t}{3\cos t}\,dt$$

$$= -\sqrt{5-4x-x^2} + \int dt$$

$t = \arcsin\frac{x+2}{3}$

$$= -\sqrt{5-4x-x^2} + t + C$$

$$= -\sqrt{5-4x-x^2} + \arcsin\left(\frac{x+2}{3}\right) + C$$

6) $\int x \cdot \arcsin x^2 \, dx = I$

$\arcsin x^2 = u \qquad x\,dx = dv$

$\frac{2x}{\sqrt{1-x^4}} dx = du \qquad \frac{x^2}{2} = v$

$$I = \frac{x^2}{2}\arcsin x^2 - \int \frac{x^2}{\cancel{2}} \cdot \frac{\cancel{2}x}{\sqrt{1-x^4}} dx$$

$$= \frac{x^2}{2}\arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}} dx$$

$$\left.\begin{aligned} 1-x^4 &= t \\ -4x^3\,dx &= dt \\ -x^3\,dx &= \frac{dt}{4} \end{aligned}\right\} \Rightarrow I = \frac{x^2}{2}\arcsin x^2 + \int \frac{dt}{4\sqrt{t}}$$

$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{2}\sqrt{t} + C$$

$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{2}\sqrt{1-x^4} + C$$

7) $\int x^3 \cdot \ln^2 x \, dx = I$

$\ln^2 x = u \qquad x^3 dx = dv \qquad \ln x = u$

$2\frac{\ln x}{x} dx = du \qquad \frac{x^4}{4} = v \qquad \frac{dx}{x} = du$

$$I = \frac{x^4}{4}\ln^2 x - \int \frac{2}{x}\ln x \cdot \frac{x^4}{4} dx$$

$$= \frac{x^4}{4}\ln^2 x - \frac{1}{2}\int x^3 \ln x \, dx$$

$$= \frac{x^4}{4}\ln^2 x - \frac{1}{2}\left[\frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{dx}{x}\right]$$

$$= \frac{x^4}{4}\ln^2 x - \frac{x^4}{8}\ln x + \frac{1}{8}\frac{x^4}{4} + C$$

8) $$\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \int \frac{(a^2\sec^2 t-a^2)^{3/2}}{\cancel{a\sec t}} \cdot \cancel{a\sec t}\,\tan t\,dt$$

$x = a\sec t$

$dx = a\sec t\,\tan t\,dt$

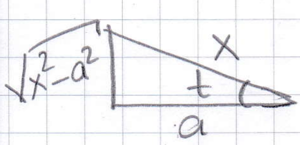
$$= \int a^3 \tan^3 t \cdot \tan t\,dt$$

$$= a^3\int \tan^2 t\,(\sec^2 t - 1)\,dt$$

$$= a^3\int \tan^2 t\,\sec^2 t\,dt - a\int \tan^2 t\,dt$$

$\tan t = u$

$\sec^2 t\,dt = du$



$$= a^3\int u^2\,du - a\int(\underbrace{\tan^2 t + 1}_{\sec^2 t} - 1)\,dt$$

$$= a^3\cdot\frac{u^3}{3} - a\int \sec^2 t\,dt + a\int dt$$

$$= \frac{a^3}{3}u^3 - a\tan t + at + c$$

$$= \frac{a^3}{3}\tan^3 t - a\tan t + at + c$$

$$= \frac{a^3}{3}\,\frac{(x^2-a^2)^{3/2}}{a^3} - a^3\cdot\frac{(x^2-a^2)^{1/2}}{a} + a^3\cdot\arccos\frac{a}{x} + c$$

$$= \frac{1}{3}(x^2-a^2)^{3/2} - a^2(x^2-a^2)^{1/2} + a^3\cdot\arccos\left(\frac{a}{x}\right) + c$$

9) $$I = \int \frac{x^4\,dx}{x^3-8} = \int \left[x + \frac{8x}{x^3-8}\right]dx$$

$$\begin{array}{r|l} x^4 & x^3-8 \\ \cline{2-2} -\cancel{x^4}+8x & x \\ \hline 8x & \end{array}$$

$$= \int x\,dx + 8\int \frac{x}{(x-2)(x^2+2x+4)}\,dx$$

$$= \frac{x^2}{2} + 8\int\left[\frac{1}{6(x-2)} + \frac{-\frac{x}{6}+\frac{1}{3}}{x^2+2x+4}\right]dx$$

$$\frac{x}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$\Rightarrow\ x \equiv A(x^2+2x+4) + (Bx+C)(x-2)$

$x \equiv (A+B)x^2 + (2A-2B+C)x + 4A-2C$

$\Rightarrow\ A+B=0,\quad 2A-2B+C=1\quad 4A-2C=0$

$$B = -A \qquad C = 2A \Rightarrow 2A + 2A + 2A = 1 \Rightarrow A = 1/6$$

$$B = -1/6$$

$$C = 1/3$$

$$\Rightarrow I = \frac{x^2}{2} + \frac{4}{3}\int \frac{dx}{x-2} - \frac{4}{3}\int \frac{2-x}{x^2+2x+4}\,dx$$

$$\left.\begin{aligned} x^2+2x+4 &= u \\ (2x+2)\,dx &= du \end{aligned}\right\} \Rightarrow I = \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{4}{3}\left(-\frac{1}{2}\right)\int \frac{-2(-x+2)dx}{x^2+2x+4}$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| + \frac{2}{3}\int \frac{2x-4+6-6}{x^2+2x+4}\,dx$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| + \frac{2}{3}\int \frac{2x+2}{x^2+2x+4} - \frac{12}{3}\int \frac{dx}{x^2+2x+4}$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| + \frac{2}{3}\ln|x^2+2x+4| - \frac{12}{3}\int \frac{dx}{(x+1)^2+3}$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| + \frac{2}{3}\ln|x^2+2x+4| - \frac{12}{3}\,\frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

10) $$\int_1^3 \frac{\sqrt{3x-x^2}}{x^3}\,dx = \int_{\sqrt{2}}^{0} t\cdot\frac{(t^2+1)^{\cancel{2}}}{9}\cdot\left(\frac{-6t}{\cancel{(t^2+1)^2}}\right)dt$$

$$\frac{\sqrt{3x-x^2}}{x} = t \qquad = \frac{2}{3}\int_0^{\sqrt{2}} t^2\,dt = \frac{2}{3}\,\frac{t^3}{3}\Big|_0^{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{9}$$

$$\Rightarrow \sqrt{3x-x^2} = tx$$

$$3x - x^2 = t^2x^2$$

$$3x = (t^2+1)\,x^2$$

$$\Rightarrow x = \frac{3}{t^2+1} \qquad x=1 \Rightarrow t = \sqrt{2}$$

$$x = 3 \Rightarrow t = 0$$

$$\Rightarrow dx = \frac{-6t\,dt}{(t^2+1)^2}$$

11) $$\int_0^{2\pi/3} \frac{d\theta}{5+4\cos\theta} = \int_0^{\sqrt{3}} \frac{\frac{2\,dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)} = 2\int_0^{\sqrt{3}} \frac{dt}{9+t^2}$$

$$\tan\frac{\theta}{2} = t$$

$$d\theta = \frac{2\,dt}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$\theta = 0 \Rightarrow t = 0$

$\theta = 2\pi/3 \Rightarrow t = \sqrt{3}$

$$= \frac{2}{3}\arctan\frac{t}{3}\Big|_0^{\sqrt{3}}$$

$$= \frac{2}{3}\left(\arctan\frac{\sqrt{3}}{3} - 0\right)$$

$$= \frac{2}{3}\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{9}$$

12) $$\int_0^4 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^4 \frac{dx}{(x-1)^2}$$

$$= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{dx}{(x-1)^2} + \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^4 \frac{dx}{(x-1)^2}$$

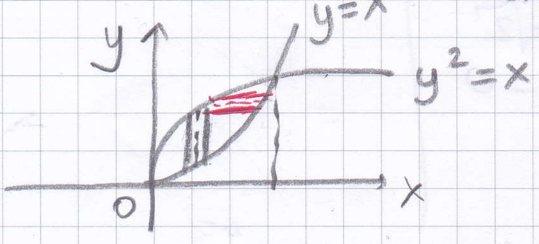
$$= \lim_{\varepsilon\to 0}\left(-\frac{1}{x-1}\Big|_0^{1-\varepsilon}\right) + \lim_{\varepsilon\to 0}\left(-\frac{1}{x-1}\Big|_{1+\varepsilon}^4\right)$$

$$= \lim_{\varepsilon\to 0}\left[\frac{-1}{-\varepsilon} - 1\right] + \lim_{\varepsilon\to 0}\left[\frac{-1}{3} + \frac{1}{\varepsilon}\right]$$

$$= \infty + \infty$$

$$= \infty$$ limit mevcut olmadığından integral mevcut değildir ıraksak-tır.

13) $y=x^2$ ile $y^2=x$ tarafından sınırlanan bölgenin alanını bulunuz.



$x^4 = x \Rightarrow x^4 - x = 0$

$x(x^3-1) = 0$

$x=0 \quad x=1$

$$\Rightarrow A = \int_0^1 (\sqrt{x} - x^2)\,dx = \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_0^1$$

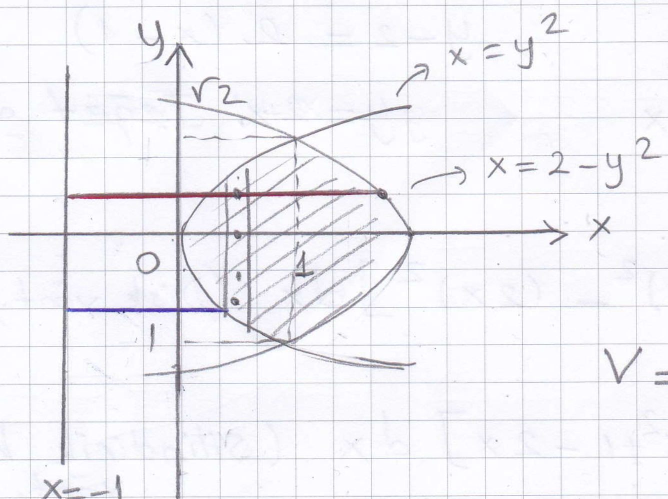
$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}\ br^2$$

(x-e göre)

1'
$$A = \int_0^1 (\sqrt{y} - y^2)\,dy = \frac{2}{3}y^{3/2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

(y'ye göre)

14) $x = y^2$, $x = 2 - y^2$ eğrileri tarafından sınırlanan bölgenin $x = -1$ doğrusu etrafında döndürülmesiyle oluşan cismin hacmini bulunuz.



$$x = 2 - x$$

$$2x = 2$$

$$x = 1 \Rightarrow y^2 = 1 \Rightarrow y = \mp 1$$

$$V = \pi \int_{-1}^{1} \left\{ \left[(2 - y^2 + 1)^2 - [y^2 + 1]^2 \right] \right\} dy$$

Disk yöntemi

$$V = 2\pi \int_0^1 (x+1) \cdot [\sqrt{x} - (-\sqrt{x})]\,dx$$

$$+ 2\pi \int_1^2 (x+1) \cdot [\sqrt{2-x} - (-\sqrt{2-x})]\,dx$$

Silindirik kabuk yöntemi

$$V = \pi \int_{-1}^{1} \left[9 - 6y^2 + \cancel{y^4} - (\cancel{y^4} + 2y^2 + 1) \right] dy$$

$$= \pi \int_{-1}^{1} (-8y^2 + 8)\, dy$$

$$= 8\pi \left(\frac{-y^3}{3} + y \right) \Big|_{-1}^{1}$$

$$= 8\pi \left[\left(\frac{-1}{3} + 1 \right) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= 8\pi \left(\frac{-2}{3} + 2 \right)$$

$$= \frac{32\pi}{3}$$

$$V = 4\pi \int_0^1 \left[x^{3/2} + x^{1/2} \right] dx$$

$$+ 4\pi \int_1^2 (x+1)\sqrt{2-x}\, dx$$

$2 - x = u^2$

$x = 2 - u^2$

$dx = -2u\,du$

$x = 1 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 0$

$$= 4\pi \left(\frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} \Big|_0^1 \right) + 4\pi \int_1^0 (2 - u^2 + 1)\, u \cdot (-2u\, du)$$

$$= \frac{8\pi}{5} + \frac{8\pi}{3} + 8\pi \int_0^1 (3 - u^2) \cdot u^2\, du$$

$$= \frac{8\pi}{5} + \frac{8\pi}{3} + 8\pi \left(u^3 - \frac{u^5}{5} \right) \Big|_0^1$$

$$= \frac{8\pi}{5} + \frac{8\pi}{3} + 8\pi \left(1 - \frac{1}{5} \right)$$

$$= \frac{8\pi}{3} + 8\pi$$

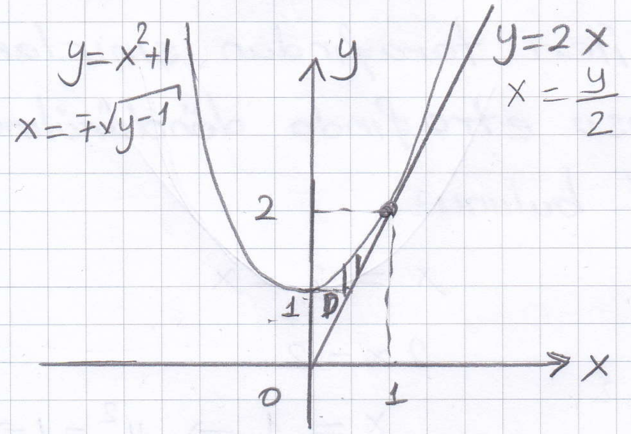
$$= \frac{32\pi}{3}$$

15) $y=x^2+1$ eğrisi ile bu eğriye $x=1$ apsisli noktasından çizilen teğet ve $x=0$ doğrusu tarafından sınırlanan bölgenin

a) Ox-ekseni

b) Oy-ekseni

etrafında döndürülmesiyle meydana gelen cismin hacmini bulunuz.



$y' = 2x$ $\qquad x=1 \Rightarrow y=2$

$y'\big|_{x=1} = 2$

$y-2 = 2.(x-1)$

$y=2x$ Teğet doğrusu

a) $V = \pi \int_0^1 [(x^2+1)^2 - (2x)^2]\,dx$ (Disk yönt.)

b) $V = 2\pi \int_0^1 x.\,[x^2+1-2x]\,dx$ (Silindirik kabuk yönt.)

$V = \pi \int_0^1 \left(\frac{y}{2}\right)^2 dy + \pi \int_1^2 \left[\left(\frac{y}{2}\right)^2 - (\sqrt{y-1})^2\right] dy$

(Disk yöntemi)