6°) $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ şeklinde ise;

Bu durumda $f_1, f_2, \ldots, f_n$ için ayrı ayrı $y_{ö1}, y_{ö2}, \ldots, y_{ön}$ özel çözümleri bulunur ve

$y_ö = y_{ö1} + y_{ö2} + \cdots + y_{ön}$ şeklinde elde edilir.

Ör/ $y'' - 5y' + 6y = e^{3x} - x^2$ dif. denkleminin genel çözümünü bulunuz.

$(D^2 - 5D + 6)y = \underbrace{e^{3x}}_{f_1} - \underbrace{x^2}_{f_2}$ $\quad$ $\alpha = 3 = r_2$

K.D: $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r_1 = 2 \quad r_2 = 3$

$$\boxed{y_h = c_1 e^{2x} + c_2 e^{3x}}$$

f_1 için ; $y_{ö1} = x.A.e^{3x}$

$y_{ö1}' = Ae^{3x} + 3xAe^{3x} = (1+3x)Ae^{3x}$

$y_{ö}'' = 3Ae^{3x} + 3(1+3x)Ae^{3x} = (6+9x)Ae^{3x}$

$\Rightarrow 6Ae^{3x} + 9xe^{3x} - 5\left[Ae^{3x} + 3Axe^{3x}\right] + 6xAe^{3x} = e^{3x}$

$6Ae^{3x} + \cancel{9xe^{3x}} - 5Ae^{3x} - \cancel{15Axe^{3x}} + 6x\cancel{Ae^{3x}} = e^{3x}$

$Ae^{3x} = e^{3x} \Rightarrow A = 1$

$$\Rightarrow \boxed{y_{ö1} = xe^{3x}}$$

$$\left.\begin{aligned} y_{ö_2} &= ax^2+bx+c \\ y'_{ö_2} &= 2ax+b \\ y''_{ö_2} &= 2a \end{aligned}\right\} \Rightarrow 2a-5(2ax+b)+6(ax^2+bx+c) \equiv -x^2$$

$$\Rightarrow 6ax^2+(-10a+6b)x+(2a-5b+6c) \equiv -x^2$$

$$6a=-1 \Rightarrow a=-1/6$$

$$-10a+6b=0 \Rightarrow b=\frac{-10}{36}$$

$$2a-5b+6c=0 \Rightarrow c=\frac{1}{6}\left[\frac{2}{6}-\frac{50}{36}\right]=\frac{-38}{6.36}=\frac{-19}{108}$$

$$\boxed{y_{ö_2}=\frac{-x^2}{6}-\frac{10x}{36}-\frac{19}{108}}$$

$$y_{G.Ç}=y_h+y_{ö_1}+y_{ö_2}=c_1e^{2x}+c_2e^{3x}+xe^{3x}-\left(\frac{x^2}{6}+\frac{10x}{36}-\frac{19}{108}\right)$$

Ör 4

$y''-6y'+8y=e^x+e^{2x}$ dif. denkleminin genel çözümünü bulunuz.

$$(D^2-6D+8)y=\underbrace{e^x}_{f_1}+\underbrace{e^{2x}}_{f_2}$$

$$y_h=c_1e^{2x}+c_2e^{4x}$$

K.D: $r^2-6r+8=0$

$(r-2)(r-4)=0$

$r_1=2 \quad r_2=4$

$$\left.\begin{aligned} y_{ö_1} &= Ae^x \\ y'_{ö_1} &= Ae^x = y''_{ö_1} = Ae^x \end{aligned}\right\} \Rightarrow Ae^x-6Ae^x+8Ae^x \equiv e^x$$

$$3Ae^x \equiv e^x \Rightarrow A=\frac{1}{3}$$

$$\boxed{y_{\ddot{o}_1} = \frac{e^x}{3}}$$

$$y_{\hat{o}_2} = A \cdot x \cdot e^{2x}$$

$$y'_{\ddot{o}_2} = Ae^{2x} + 2Axe^{2x}$$

$$y''_{\ddot{o}_2} = 3Ae^{2x} + 4Axe^{2x}$$

$$\Rightarrow 3Ae^{2x} + 4Axe^{2x} - 6\left(Ae^{2x} + 2Axe^{2x}\right) + 8Axe^{2x} \equiv e^{2x}$$

$$3Ae^{2x} \equiv e^{2x} \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$$

$$\boxed{y_{\ddot{o}_2} = \frac{xe^{2x}}{3}}$$

$$y_{G.Ç} = y_h + y_{\hat{o}_1} + y_{\ddot{o}_2} = c_1 e^{2x} + c_2 e^{4x} + \frac{e^x}{3} + \frac{xe^{2x}}{3}$$

Ör: $y^{(4)} + 8y'' - 9y = t^2 + \sin 2t$ dif. denkleminin genel çözümünü bulunuz.

$$(D^4 + 8D^2 - 9)y = t^2 + \sin 2t$$

K.D: $r^4 + 8r^2 - 9 = 0 \Rightarrow (r-1)(r^3 + r^2 + 9r + 9) = 0$

$$\Rightarrow (r-1)\left[r^2(r+1) + 9(r+1)\right] = 0$$

$$\Rightarrow (r-1)(r+1)(r^2+9) = 0$$

$$r_1 = 1 \quad r_2 = -1 \quad r_{3,4} = \mp 3i$$

$$\begin{array}{l|l} r^4 + 8r^2 - 9 & r-1 \\ \underline{-\cancel{r^4} + r^3} & r^3 + r^2 + 9r + 9 \\ \cancel{r^3} + 8r^2 - 9 & \\ \underline{-\cancel{r^3} + r^2} & \\ \cancel{9r^2} - 9 & \\ \underline{-\cancel{9r^2} + 9r} & \\ 9r - 9 & \\ \underline{9r - 9} & \\ 0 & \end{array}$$

$$\boxed{y_h = c_1 e^{t} + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t}$$

$$\left.\begin{aligned} y_{\ddot{o}_1} &= at^2 + bt + c \\ y'_{\ddot{o}_1} &= 2at + b \\ y''_{\ddot{o}_1} &= 2a \\ y'''_{\ddot{o}_1} &= y^{(4)} = 0 \end{aligned}\right\} \quad 0 + 8 \cdot 2a - 9(at^2 + bt + c) \equiv t^2$$

$$-9at^2 - 9bt + (16a - 9c) \equiv t^2$$

$$\left.\begin{aligned} -9a &= 1 \quad \Rightarrow \quad a = -1/9 \\ 9b &= 0 \quad \Rightarrow \quad b = 0 \\ 16a - 9c &= 0 \quad \Rightarrow \quad c = -\frac{16}{81} \end{aligned}\right\} \Rightarrow \boxed{y_{\ddot{o}_1} = -\frac{t^2}{9} - \frac{16}{81}}$$

$$\left.\begin{aligned} y_{\ddot{o}_2} &= A \sin 2t + B \cos 2t \\ y'_{\ddot{o}_2} &= 2A \cos 2t - 2B \sin 2t \\ y''_{\ddot{o}_2} &= -4A \sin 2t - 4B \cos 2t \\ y'''_{\ddot{o}_2} &= -8A \cos 2t + 8B \sin 2t \\ y^{(4)}_{\ddot{o}_2} &= 16A \sin 2t + 16B \cos 2t \end{aligned}\right\} \quad 16A \sin 2t + 16B \cos 2t + 8(-4A \sin 2t - 4B \cos 2t) - 9(A \sin 2t + B \cos 2t)$$

$$\equiv \sin 2t$$

$$-25A \sin 2t - 25B \cos 2t \equiv \sin 2t$$

$$\left.\begin{aligned} -25A &= 1 \\ -25B &= 0 \end{aligned}\right\} \quad A = -1/25, \quad B = 0$$

$$\boxed{y_{\ddot{o}_2} = -\frac{\sin 2t}{25}}$$

$$y_{G.Ç} = y_h + y_{ö1} + y_{ö2}$$

$$= c_1 e^{t} + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{t^2}{9} - \frac{16}{81} - \frac{1}{25} \sin 2t$$

Ör/ $\underbrace{y''' + y'' = x^2 + 1}_{f_1} + \underbrace{3xe^{x}}_{f_2}$ dif. denkleminin genel çözümünü bulunuz.

$$(D^3 + D^2) y = x^2 + 1 + 3xe^{x}$$

K.D: $r^3 + r^2 = 0$

$r^2 (r+1) = 0$

$r_{1,2} = 0 \quad r_3 = -1$

$$\boxed{y_h = c_1 + c_2 x + c_3 e^{-x}}$$

$$y_{ö1} = x^2 (ax^2 + bx + c) = ax^4 + bx^3 + cx^2$$

$$\left.\begin{aligned} y_{ö1}' &= 4ax^3 + 3bx^2 + 2cx \\ y_{ö1}'' &= 12ax^2 + 6bx + 2c \\ y_{ö1}''' &= 24ax + 6b \end{aligned}\right\} \Rightarrow 24ax + 6b + 12ax^2 + 6bx + 2c \equiv x^2 + 1$$

$$12ax^2 + (24a + 6b)x + (6b + 2c) \equiv x^2 + 1$$

$12a = 1 \quad \Rightarrow a = 1/12$

$24a + 6b = 0 \Rightarrow b = -1/3$

$6b + 2c = 1 \Rightarrow c = \frac{3}{2}$

$$\boxed{y_{ö1} = \frac{x^4}{12} - \frac{x^3}{3} + \frac{3x^2}{2}}$$

$$y_{\hat{\ddot{o}}2} = (ax+b).e^x$$

$$y'_{\ddot{o}2} = ae^x + (ax+b)e^x$$

$$y''_{\hat{\ddot{o}}2} = ae^x + ae^x + (ax+b)e^x = 2ae^x + (ax+b)e^x$$

$$y'''_{\ddot{o}2} = 2ae^x + ae^x + (ax+b)e^x = 3ae^x + (ax+b)e^x$$

$$\Rightarrow 3ae^x + (ax+b)e^x + 2ae^x + (ax+b)e^x \equiv 3xe^x$$

$$\Rightarrow 2ax\,e^x + (5a+2b)e^x \equiv 3xe^x$$

$$\left.\begin{array}{l} 2a = 3 \Rightarrow a = 3/2 \\ 5a+2b = 0 \Rightarrow b = -\frac{15}{4} \end{array}\right\} \boxed{y_{\hat{\ddot{o}}2} = \left(\frac{3x}{2} - \frac{15}{4}\right)e^x}$$

$$y_{G.Ç} = c_1 + c_2 x + c_3 e^{-x} + \frac{x^4}{12} - \frac{x^3}{3} + \frac{3x^2}{2} + \left(\frac{3x}{2} - \frac{15}{4}\right)e^x$$

Sabitlerin Değişimi Yöntemi -

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x) \qquad (a_0 \neq 0)$$

Bu n. mertebeden dif. denklemin ikinci tarafsız kısmının genel çözümü, $c_1, c_2, \ldots, c_n$ ler keyfi parametreler ve $y_1, y_2, \ldots, y_n$ ikinci tarafsız denklemin lineer bağımsız çözüm takımı olmak üzere

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

şeklindedir. İkinci taraflı denklemin genel çözümünü bu elde ettiğimiz çözümden yararlanarak bulup bulamayacağımızı araştıralım. Bunun için $c_1, c_2, \ldots, c_n$ keyfi parametrelerinin x bağımsız değişkeninin bir fonksiyonu olduklarını varsayalım. Yani,

$c_1 = c_1(x)$, $c_2 = c_2(x)$, ..., $c_n = c_n(x)$ olsun.

$$y = c_1(x)y_1 + c_2(x)y_2 + \cdots + c_n(x)y_n$$

$$y' = \underbrace{c_1'y_1 + c_2'y_2 + \cdots + c_n'y_n}_{\substack{\text{keyfi olarak}\\ =0 \text{ alalım.}}} + c_1y_1' + c_2y_2' + \cdots + c_ny_n' \qquad (1)$$

$$y' = c_1y_1' + c_2y_2' + \cdots + c_ny_n'$$

$$y'' = \underbrace{c_1'y_1' + c_2'y_2' + \cdots + c_n'y_n'}_{=0 \text{ (keyfi olarak)}} + c_1y_1'' + c_2y_2'' + \cdots + c_ny_n'' \qquad (2)$$

$$y'' = c_1y_1'' + c_2y_2'' + \cdots + c_ny_n''$$

$$\vdots$$

$$y^{(n-1)} = \underbrace{c_1'y_1^{(n-2)} + c_2'y_2^{(n-2)} + \cdots + c_n'y_n^{(n-2)}}_{=0 \text{ (keyfi olarak)}} + c_1y_1^{(n-1)} + \cdots + c_ny_n^{(n-1)} \qquad (n-1)$$

$$y^{(n-1)} = c_1y_1^{(n-1)} + \cdots + c_ny_n^{(n-1)}$$

$$y^{(n)} = c_1'y_1^{(n-1)} + \cdots + c_n'y_n^{(n-1)} + c_1y_1^{(n)} + \cdots + c_ny_n^{(n)}$$

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-2} y'' + a_{n-1} y' + a_n y = f(x)$$

$$a_0 \left[c_1' y_1^{(n-1)} + \dots + c_n' y_n^{(n-1)} + c_1 y_1^{(n)} + \dots + c_n y_n^{(n)} \right] + a_1 \left[c_1 y_1^{(n-1)} + \dots + c_n y_n^{(n-1)} \right] + \dots + a_{n-2} \left[c_1 y_1'' + c_2 y_2'' + \dots + c_n y_n'' \right]$$

$$+ a_{n-1} \left[c_1 y_1' + c_2 y_2' + \dots + c_n y_n' \right] + a_n \left[c_1(x) y_1 + c_2(x) y_2 + \dots + c_n(x) y_n \right] = f(x)$$

$$a_0 \left[c_1' y_1^{(n-1)} + \dots + c_n' y_n^{(n-1)} \right] + c_1 \left[\overbrace{a_0 y_1^{(n)} + a_1 y_1^{(n-1)} + \dots + a_{n-2} y_1'' + a_{n-1} y_1' + a_n y_1}^{=0} \right]$$

$$+ c_2 \left[\overbrace{a_0 y_2^{(n)} + a_1 y_2^{(n-1)} + \dots + a_{n-2} y_2'' + a_{n-1} y_2' + a_n y_2}^{=0} \right]$$

$$+ \dots + c_n \left[\overbrace{a_0 y_n^{(n)} + a_1 y_n^{(n-1)} + \dots + a_{n-2} y_n'' + a_{n-1} y_n' + a_n y_n}^{=0} \right] = f(x)$$

$$a_0 \left[c_1' y_1^{(n-1)} + \dots + c_n' y_n^{(n-1)} \right] = f(x) \Rightarrow \underline{c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \dots + c_n' y_n^{(n-1)} = \frac{f(x)}{a_0}}$$

(n)

$$\left.\begin{array}{l} \text{n-1 tane}\left\{\begin{array}{l} c_1' y_1 + c_2' y_2 + \cdots + c_n' y_n = 0 \\ c_1' y_1' + c_2' y_2' + \cdots + c_n' y_n' = 0 \\ \vdots \\ c_1' y_1^{(n-2)} + c_2' y_2^{(n-2)} + \cdots + c_n' y_n^{(n-2)} = 0 \end{array}\right. \\ \text{n. denk.}\left\{ c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \cdots + c_n' y_n^{(n-1)} = \dfrac{f(x)}{a_0} \right. \end{array}\right\} \rightarrow$$

Dolayısıyla sistem Cramer sistemidir.

$y_1, y_2, \ldots, y_n$'ler Lineer bağımsız çözümler olduklarından sistemin katsayılar matrisinin determinantı sıfırdan farklıdır.

$$c_i' = \frac{\Delta_i}{\Delta} \Rightarrow c_i = \int \frac{\Delta_i}{\Delta}\, dx + k_i$$

$$\Delta = \begin{vmatrix} y_1 & y_2 & \cdots & y_i & \cdots & y_n \\ y_1' & y_2' & \cdots & y_i' & \cdots & y_n' \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_i^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$\Delta_i = \begin{vmatrix} y_1 & y_2 & \cdots & 0 & \cdots & y_n \\ y_1' & y_2' & \cdots & 0 & \cdots & y_n' \\ \vdots & & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \dfrac{f(x)}{a_0} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Bulunan c_i'ler $y=c_1y_1+\dots+c_ny_n$'de yerine yazılmak suretiyle istenen genel çözüm hem ikinci tarafsız denklemin genel çözümünü hemde ikinci taraflı denklemin özel çözümünü içerecek şekilde elde edilir.

Yöntemin belirsiz katsayılar yönteminde ele alınmayan $f(x)$ fonksiyonları için uygulanması kolaylık sağlar. ($\ln x$, $\sec x$, $\tan x$, $\csc x$, $\cot x$)

Ör/ $y''+y=\tan x$ $\quad \left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right)$ denkleminin genel çözümünü sabitlerin (parametrelerin) değişimi yöntemi ile bulunuz.

$(D^2+1)\,y=\tan x$

K.D: $r^2+1=0$

$r_{1,2}=\mp i$

$y=c_1\cos x+c_2\sin x$

$c_1=c_1(x)\ ,\quad c_2=c_2(x)$

$y=c_1(x)\cos x+c_2(x)\sin x$

$$\begin{aligned}\sin x\ /\ & c_1'\cos x+c_2'\sin x=0\\ \cos x\ /\ & -c_1'\sin x+c_2'\cos x=\tan x\end{aligned}$$

$$c_2'\sin^2 x+c_2'\cos^2 x=\sin x$$

$$c_2'=\sin x$$

$$c_2=\int \sin x\,dx+k_2=-\cos x+k_2$$

$$\Rightarrow c_1' \cos x + \sin^2 x = 0 \Rightarrow c_1' = \frac{-\sin^2 x}{\cos x} \Rightarrow c_1 = -\int \frac{\sin^2 x}{\cos x}\,dx + k_1$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx + k_1$$

$$= -\int \frac{1}{\cos x}\,dx + \int \cos x\,dx + k_1$$

$$= -\ln|\sec x + \tan x| + \sin x + k_1$$

$$\Rightarrow y = \left[-\ln|\sec x + \tan x| + \sin x + k_1\right]\cos x + \left[-\cos x + k_2\right].\sin x$$

$$= -\cos x . \ln|\sec x + \tan x| + \cancel{\sin x \cos x} + k_1 \cos x - \cancel{\sin x \cos x} + k_2 \sin x$$

$$= \underbrace{k_1 \cos x + k_2 \sin x}_{y_h} \underbrace{- \cos x . \ln|\sec x + \tan x|}_{y_{ö}}$$