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Parametreye Bağlı İntegraller

Eğer İntegrand, integral değişkenine ek olarak bir veya daha fazla parametrenin fonksiyonu ise

örneğin $\int_0^{\pi/2} \sin \alpha x \, dx = -\frac{1}{\alpha} \cos \alpha x \Big|_0^{\pi/2} = \frac{1}{\alpha} - \frac{1}{\alpha} \cos \frac{\alpha \pi}{2} = F(\alpha)$

$$\int_1^2 (x+\alpha)^2 dx = \frac{(x+\alpha)^3}{3} \Big|_1^2 = \frac{(2+\alpha)^3}{3} - \frac{(1+\alpha)^3}{3} = G(\alpha)$$

örneklerinde görüldüğü gibi integral bu parametlerin bir fonksiyonudur. Böylece

$$\int_a^b f(x,\alpha) dx = F(\alpha)$$

$$\int_a^b f(x,\alpha,\beta) = G(\alpha,\beta)$$

integral sınırları a, b sabit veya parametreye bağlı fonksiyonlar olmak üzere bazen $f(x,\alpha)$ öyleki integral değeri çok komplike veya imkansız olabilir. Ancak $\frac{\partial F}{\partial \alpha}$ integrandlı integral daha kolay değerlendiri-lebilir.

İntegral İşareti Altında Türev (Leibnitz Kuralı)

Teorem: $a \leq x \leq b$, $c \leq \alpha \leq d$, a, b α dan bağımsız olmak üzere $f(x,\alpha)$, $\frac{\partial f}{\partial \alpha}$ x ve α nın sürekli fonksiyonları ise

$$\frac{d}{d\alpha} \int_a^b f(x,\alpha) dx = \int_a^b \frac{df(x,\alpha)}{d\alpha} dx$$

Kanıt

$$F(\alpha) = \int_a^b f(x,\alpha)dx \qquad (*)$$

$$F(\alpha+\Delta\alpha) = \int_a^b f(x, \alpha+\Delta\alpha)\, dx \quad (**)$$

$(**) - (*) \Rightarrow F(\alpha+\Delta\alpha) - F(\alpha) = \int_a^b [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\, dx \quad (***)$

Lagrange'nin aradeğer teo. (veya ortalama değer teo. göre)

$$f(x,\alpha+\Delta\alpha) - f(x,\alpha) = \Delta\alpha \frac{\partial f}{\partial\alpha}(x, \alpha+\theta\Delta\alpha) \quad , 0<\theta<1$$

$$F(\alpha+\Delta\alpha) - F(\alpha) = \int_a^b \Delta\alpha \frac{\partial f}{\partial\alpha}(x, \alpha+\theta\Delta\alpha)dx$$

$$\frac{F(\alpha+\Delta\alpha) - F(\alpha)}{\Delta\alpha} = \int_a^b \frac{\partial f}{\partial\alpha}(x, \alpha+\theta\Delta\alpha)dx$$

$\Delta\alpha \to 0$

$$F'(\alpha) = \frac{dF(\alpha)}{d\alpha} = \lim_{\Delta\alpha\to 0} \int_a^b \frac{\partial f}{\partial\alpha}\, dx$$

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}dx$$

Not

Böylece Leibnitz Kuralı

$$F(\alpha) = \int_a^b f(x,\alpha)dx \Rightarrow F'(\alpha) = \int_a^b \frac{df(x,\alpha)}{d\alpha}dx$$

Benzer olarak,

$$F(\alpha,\beta) = \int_a^b f(x,\alpha,\beta)\, dx \Rightarrow \frac{\partial F}{\partial\alpha} = \int_a^b \frac{\partial f(x,\alpha,\beta)}{\partial\alpha}$$

$$\frac{\partial F(\alpha,\beta)}{\partial\beta} = \int_a^b \frac{\partial f(x,\alpha,\beta)}{\partial\beta}dx$$

2) **NOT**

Eğer f fonksiyonu

a.) $[a,b]$ üzerinde sürekliyse $\exists$ bazı $c \in (a,b)$ olacak şekilde

$$\int_a^b f(x)dx = (b-a)f(c)$$

b.) $[a, a+h]$ da sürekliyse $\exists$ $\theta \in (0,1)$ reel sayı olacak şekilde

$$\int_a^{a+h} f(x)dx = h\, f(a+\theta h)$$

Teorem

Eğer $f(x,\alpha)$ ve $\frac{\partial f(x,\alpha)}{\partial \alpha}$ x ve α nın sürekli fonksiyonları $a \le x \le b$ $c \le \alpha \le d$ ve a, b α parametresinin fonksiyonları 1. mertebeden türevleri de sürekli ise

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, d\alpha = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx + \frac{db(\alpha)}{d\alpha} f(b,\alpha) - \frac{da}{d\alpha} f(a,\alpha)$$

$$\begin{cases} a = a(\alpha) \\ b = b(\alpha) \end{cases}$$

Kanıt

$$F(\alpha) = \int_a^b f(x,\alpha)dx \quad (*)$$

$$F(\alpha+\Delta\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x, \alpha+\Delta\alpha)\, dx \quad (**)$$

$$\begin{cases} \alpha \to \alpha+\Delta\alpha \\ b \to b+\Delta b \\ a \to a+\Delta a \end{cases}$$

$(**)-(*)$; $F(\alpha+\Delta\alpha) - F(\alpha)$

$$= \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx - \int_a^b f(x,\alpha)dx + \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx - \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx$$

$$= \int_{a+\Delta a}^{b+\Delta b} [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]dx + \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx - \int_a^b f(x,\alpha)dx$$

$$\left\{ \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)\,dx = \int_{a+\Delta a}^{b} f(x,\alpha)\,dx + \int_{b}^{b+\Delta b} f(x,\alpha)\,dx = -\int_{b}^{a+\Delta a} f(x,\alpha)\,dx + \int_{b}^{b+\Delta b} f(x,\alpha)\,dx \right\}$$

$$= \int_{a+\Delta a}^{b+\Delta b} \Delta\alpha \frac{\partial f}{\partial \alpha}(x, \alpha+\theta\Delta\alpha)\,dx + \int_{b}^{b+\Delta b} f(x,\alpha) - \Big(\underbrace{\int_{a}^{b} f(x,\alpha) + \int_{b}^{a+\Delta a} f(x,\alpha)\,dx}_{\int_{a}^{a+\Delta a} f(x,\alpha)\,dx}$$

$$F(\alpha+\Delta\alpha) - F(\alpha) = \int_{a+\Delta a}^{b+\Delta b} \Delta\alpha \frac{\partial f}{\partial \alpha}(x, \alpha+\theta\Delta\alpha)\,dx + \Delta b\, f(b+\theta_1\Delta b, \alpha) - \Delta a\, f(a+\theta_2\Delta a, \alpha)$$

$$\frac{F(\alpha+\Delta\alpha) - F(\alpha)}{\Delta\alpha} = \int_{a+\Delta a}^{b+\Delta b} \frac{\partial f}{\partial \alpha}(x, \alpha+\theta\Delta\alpha)\,dx + \frac{\Delta b}{\Delta\alpha} f(b+\theta_1\Delta b, \alpha) - \frac{\Delta a}{\Delta\alpha} f(a+\theta_2\Delta a, \alpha)$$

$0 < \theta_1, \theta_2, \theta < 1$ ve

$\Delta\alpha \to 0$, $\Delta a \to 0$ ve $\Delta b \to 0$

$$\frac{dF(\alpha)}{d\alpha} = F'(\alpha) = \lim_{\Delta\alpha\to 0} \frac{\Delta a}{\Delta\alpha} \underbrace{\lim_{\Delta a\to 0} f(a+\theta_2\Delta a, \alpha)}_{f(a,\alpha)} + \lim_{\Delta\alpha\to 0} \frac{\Delta b}{\Delta\alpha} \underbrace{\lim}_{f(b,\alpha)}$$

$$\Rightarrow \frac{d}{d\alpha} \int_{a}^{b} f(x,\alpha)\,dx = \int_{a}^{b} \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx + \frac{db}{d\alpha} f(b,\alpha) - \frac{da}{d\alpha} f(a,\alpha)$$

③ Ex:

1) $\frac{d}{dx}\int_{\frac{\pi}{4}}^{x} \sin t\,dt = \frac{d}{dx}\left(-\cos t \Big|_{\frac{\pi}{4}}^{x}\right) = \frac{d}{dx}\left(-\cos x + \cos\frac{\pi}{4}\right)$

$= \underline{\underline{\sin x}}$

2. Metod Leibnitz kuralı ile

$a = \frac{\pi}{4}$, $b = x$, $\alpha = x$

$$\frac{d}{dx}\int_{\frac{\pi}{4}}^{x} \sin t\,dt = \int_{\frac{\pi}{4}}^{x} \frac{\partial}{\partial x}(\sin t)\,dt + \underbrace{\frac{dx}{dx}}_{1}\sin x - \underbrace{\frac{d(\frac{\pi}{4})}{dx}}_{0}\sin\frac{\pi}{4}$$

$= \sin x$

2.) $I = \int_{x^2}^{\sin^{-1}x} \frac{\sin t}{t}\,dt \rightarrow \frac{dI}{dx} = ?$

3.) $\int_0^{\infty} t^n e^{-kt^2}\,dt$ n tek, $k>0$

$$\frac{dI}{dx} = \frac{d}{dx}\int_{x^2}^{\sin^{-1}x} \frac{\sin t}{t}\,dt = \underbrace{\int_{x^2}^{\sin^{-1}x} \frac{\partial}{\partial x}\left(\frac{\sin t}{t}\right)dt}_{0} + \frac{d(\sin^{-1}x)}{dx}\,\frac{\sin(\sin^{-1}x)}{\sin^{-1}x}$$

$$- \frac{\partial(x^2)}{\partial x}\,\frac{\sin(x^2)}{x^2}$$

$$= \frac{1}{\sqrt{1-x^2}}\,\frac{x}{\sin^{-1}x} - 2x\,\frac{\sin(x^2)}{x^2}$$

3.) $n=1 \rightarrow \int_0^{\infty} t\,\underbrace{e^{-kt^2}}\,dt = \int_0^{\infty} e^{u}\,\frac{du}{-2k} = \frac{-1}{2k}\,e^{u}\Big|_{t=0}^{\infty}$

$-kt^2 = u$

$-2kt\,dt = du$

$$= -\frac{1}{2k}\,e^{-kt^2}\Big|_0^{\infty} = \frac{1}{2k}$$

$\Rightarrow$ k ya göre her iki tarafın ~~...~~ türevi

$$\frac{d}{dk}\left(\int_0^\infty t e^{-kt^2} dt\right) = \frac{d}{dk}\left(\frac{1}{2k}\right)$$

$$\int_0^\infty \frac{\partial}{\partial k}\left(t e^{-kt^2}\right) dt = \frac{d}{dk}\left(\frac{1}{2k}\right)$$

$$\Rightarrow -\int_0^\infty t^3 e^{-kt^2} dt = -\frac{1}{2k^2}$$

$$\int_0^\infty t^3 e^{-kt^2} dt = \frac{1}{2k^2}$$ (tekrar k ya göre türev)

$$\Rightarrow \frac{d}{dk}\int_0^\infty t^3 e^{-kt^2} dt = \frac{d}{dk}\left(\frac{1}{2k^2}\right)$$

$$-\int_0^\infty t^5 e^{-kt^2} dt = -\frac{1}{k^3}$$ (tekrak k ya göre türev.

$$\Rightarrow \int_0^\infty t^7 e^{-kt^2} dt = \frac{3}{k^4}$$

$\vdots$

$$\int t^{2n+1} e^{-kt^2} dt = \frac{n!}{2k^{n+1}} \quad , n = 0, 1, 2, \ldots$$

(4)

örnek

$$\int_0^1 \ln(x^2+y^2)dy = I(x)$$

$$\frac{dI}{dx} = \frac{d}{dx}\int_0^1 \ln(x^2+y^2)\,dy = \int_0^1 \frac{\partial}{\partial x}\left[\ln(x^2+y^2)\,dy\right] = \int_0^1 \frac{2x}{x^2+y^2}dy$$

$$\Rightarrow \frac{2x}{x^2}\int_0^1 \frac{dy}{1+\left(\frac{y}{x}\right)^2} \quad , \quad \frac{y}{x} = u$$

$$dy = x\,du$$

$$\Rightarrow \frac{2x}{x^2}\int \frac{x\,du}{1+u^2} = 2\int \frac{du}{1+u^2} = 2\,\mathrm{Arctan}\,u$$

$$\Rightarrow \frac{dI}{dx} = 2\arctan\frac{y}{x}\Big|_{y=0}^{1} = 2\,\mathrm{Arctan}\left(\frac{1}{x}\right) - 2\arctan 0$$

$$= 2\arctan\frac{1}{x}$$

x'e göre int. al.

$$I(x) = 2\int \mathrm{Arctan}\left(\frac{1}{x}\right)dx + C$$

$$u = \arctan\frac{1}{x}$$

$$du = \frac{-1/x^2}{1+1/x^2}dx = \frac{-dx}{1+x^2}$$

$$dx = dv$$

$$x = v$$

$$I = 2\left(x\,\mathrm{Arctan}\left(\frac{1}{x}\right) + \int \frac{x\,dx}{1+x^2}\right) + C$$

for $x=0$

$$= 2x\arctan\left(\frac{1}{x}\right) + \ln(1+x^2) + C$$

$$I(0) = \int_0^1 \ln(y^2)dy = 2\int_0^1 \ln y\,dy$$

$$= 2\lim_{a\to 0}\int_a^1 \ln y\,dy = 2\lim_{a\to 0}(-a\ln a - 1)$$

$$= -2 \qquad \boxed{C = -2}$$

$$I(0) = \underbrace{2.0 \arctan\left(\tfrac{1}{0}\right)}_{0} + \underbrace{\ln(1)}_{0} + C = C$$

Duration: 70 Min.

$$\Rightarrow I(x) = 2x \arctan\left(\frac{1}{x}\right) + \ln(1+x^2) - 2$$

ödev : $\int_0^{\infty} t^n e^{-at^2} dt$, $n = 2, 4, \dots 2m \Rightarrow$

$$\int_0^{\infty} e^{-at^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

(5) örnek

$$y = \int_0^x f(u)\sin(x-u)\,du$$ nun $y'' + y = f(x)$

sağladığını gösterin.

$$y' = \frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}\left[f(u)\cdot\sin(x-u)\,du\right] + \underbrace{\underbrace{\frac{dx}{dx}}_{1} f(x)\sin 0}_{0} - \underbrace{\frac{d(0)}{dx}}_{0} f(0)\sin x$$

$$\int_0^x \frac{\partial}{\partial x}\left(f(u)\sin(x-u)\,du\right)$$

$$y' = \int_0^x f(u)\cos(x-u)\,du \checkmark$$

$$y'' = \frac{dy'}{dx} = \int_0^x \frac{\partial}{\partial x}\left[f(u)\cos(x-u)\,du\right] + \underbrace{\frac{d(x)}{dx}}_{1} f(x)\underbrace{\cos 0}_{1} - \frac{d(0)}{dx}\underbrace{f(0)\cos x}_{0}$$

$$= \int_0^x \frac{\partial}{\partial x}\left[f(u)\cos(x-u)\,du\right] + f(x)$$

$$y'' = -\int_0^x f(u)\sin(x-u)\,du + f(x)$$

$$y = \int_0^x f(u)\sin(x-u)\,du$$

$$y'' + y = f(x)$$

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