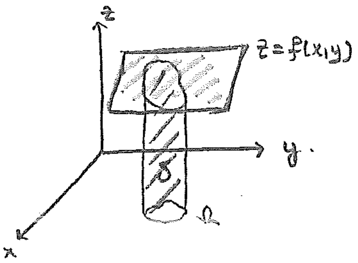
KATLI INTEGRALLER

İki Katlı (Double) İntegraller

S üç boyutlu bölgesi üstten $z=f(x,y)$ yüzeyi alttan xy-düzlemi yandan ise Ω'nın sınırlarından geçen ve z-eksenine paralel olan silindirik yüzey ile sınırlı olsun. S bölgesinin hacmini hesaplayalım.

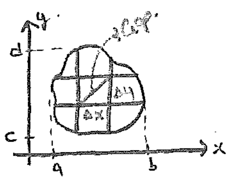


Ω: $a \le x \le b$; $c \le y \le d$ şeklinde dikdörtgensel bölge f ise Ω'da tanımlı sınırlı bir fonksiyon olsun. Ω bölgesini koordinat eksenlerine paralel doğrular ile alt dikdörtgensel bölgelere ayıralım.

$$a = x_0 < x_1 < x_2 < \dots < x_m = b$$
$$c = y_0 < y_1 < y_2 < \dots < y_n = d$$

Bu taktirde Ω bölgesi $m \times n$ tane R_{ij} ($1 \le i \le m$, $1 \le j \le n$) dikdörtgeninden oluşur. R_{ij} dikdörtgeninin alanı;

$$\Delta A_{ij} = \Delta x_i \cdot \Delta y_j = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$$ şeklinde yazılır.



Her bir R_{ij} dikdörtgeninin içinde keyfi $(\bar{x}_i, \bar{y}_j)$ noktasını alalım. Bu taktirde o dikdörtgeni içeren bölgenin hacmi;

$$\boxed{\Delta V_{ij} = f(\bar{x}_i, \bar{y}_j) \cdot \Delta A_{ij}}$$ şeklinde yazılır.

Riemann Toplamı!

Hacmin yaklaşık değeri;
$$\boxed{R(f,P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j) \cdot \Delta x_i \cdot \Delta y_j}$$
şeklinde bulunabilir.

$$\boxed{\begin{gathered} Çap(R_{ij}) = \sqrt{\Delta x_i^2 + \Delta y_j^2} \\ \|P\| = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} Çap(R_{ij}) \end{gathered}}$$

$m, n \to$ sonsuza
$|\Delta x_i|, |\Delta y_j| \to$ sıfıra
şeklinde de yazabiliriz.
Ama çaptan yaklaşmak daha geneldir.

$$\boxed{\text{Eğer } \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j) \cdot \Delta A_{ij} \qquad \iint_{\Omega} f(x,y) \, dA}$$

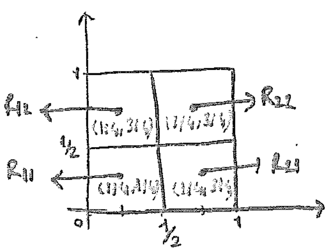
mevcut ise bu değer istenen hacmin gerçek değeridir. Ayrıca bu değere $f(x,y)$ fonksiyonunun Ω bölgesi üzerinde iki katlı integrali denir ve sembolik olarak yandaki gibi gösterilir.

** $f(x,y)=1$ ise Ω bölgesinin alanı $\iint_{\Omega} dA$ olur. $dA = dx\,dy\,(dy\,dx)$'tir.

Örnek: Ω: $0 \le x \le 1$, $0 \le y \le 1$ Ω bölgesini 4 tane alt kareye ayırarak ve noktaları herbir karenin merkezinde seçerek;

$\iint_{\Omega} (x^2+y)\,dA$ integralini yaklaşık olarak hesaplayınız.

$$f(x,y) = x^2 + y$$



$$\iint f(x,y)\,dA \approx \frac{1}{4} \cdot \left[f(1/4,1/4) + f(1/4,3/4) + f(3/4,3/4) + f(3/4,1/4) \right]$$

$dA = dx \cdot dy$

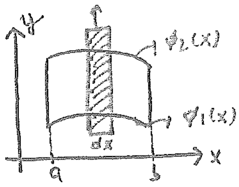
$dA = \frac{1}{2} \cdot \frac{1}{2}$

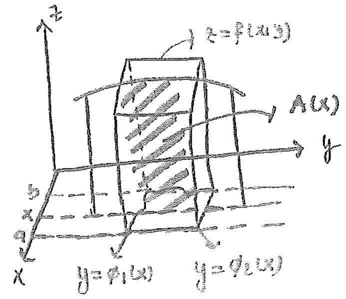
$dA = \frac{1}{4}$

$$\boxed{\iint_{\Omega} (x^2+y)\,dA \approx \frac{13}{16}}$$

İki katlı İntegralin Ardışık olarak Hesaplanması

① $R_{xy}: \ a \le x \le b \ , \ \phi_1(x) \le y \le \phi_2(x)$;





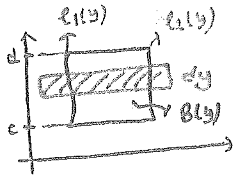
$$A(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y).dy$$

$$\int_a^b A(x).dx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y).dy.dx.$$

Eğer f(x) fonksiyonu ve bölgesinde sürekli ise;

$$\boxed{\iint_{R_{xy}} f(x,y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y).\,dy.dx}$$

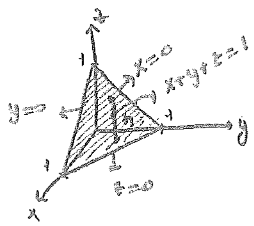
② $R_{xy}: \ c \le y \le d \ , \ \ell_1(y) \le x \le \ell_2(y)$



$$B(y) = \int_{\ell_1(y)}^{\ell_2(y)} f(x,y).dx$$

$$\boxed{\iint_{R_{xy}} f(x,y).dA = \int_c^d \int_{\ell_1(y)}^{\ell_2(y)} f(x,y).dx.dy}$$

Örnek: $x+y+z=1$ düzlemi ve koordinat düzlemleri ile sınırlı bölgenin hacmini hesaplayınız.



$x=0 \rightarrow$ yz- düzlemi

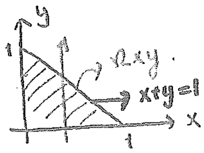
$y=0 \rightarrow$ xz- düzlemi

$z=0 \rightarrow$ xy- düzlemi

" Elimizde bölge 4 tane yüzey tarafından sınırlandırılmıştır. Bunlar $x=0$, $y=0$ $z=0$ ve $x+y+z=1$ yüzeyleridir. "

" $x+y+z=1$'in ilk 1/8'lik kısmının hacmini hesaplayın dese de aynı şey olacaktı. "

$$V = \iint_{R_{xy}} (\underbrace{1-x-y}_{z_2} - \underbrace{0}_{z_1}).dA$$



$$= \iint_{R_{xy}} (1-x-y).dA \rightarrow$$

$$V = \int_0^1 \int_0^{1-x} (1-x-y).dy.dx$$

$$= \int_0^1 (y - xy - y^2/2)\Big|_0^{1-x} .dx$$

$$= \int_0^1 [(1-x) - x(1-x) - (1-x)^2/2].dx$$

$$= \int_0^1 (1-x)\left(1-x-\frac{(1-x)}{2}\right).dx$$

$$= \frac{1}{2}\int_0^1 (1-x).(1-x).dx = \frac{1}{2}\int_0^1 (1-x)^2.dx.$$

$$= -\frac{1}{2}\left[\frac{(1-x)^3}{3}\right]\Big|_0^1 = +\frac{1}{2}\left(\frac{1}{3}\right) \rightarrow \boxed{V = \frac{1}{6}}$$

Örnek: $\int_1^2 \int_x^{x^2} (x+y).dy.dx$

$$\int_1^2 (xy + y^2/2)\Big|_x^{x^2} dx = \int_1^2 [(x^3 + x^4/2 - (x^2 + x^2/2)].dx$$

$$\frac{x^4}{4} + \frac{x^5}{10} - \frac{x^3}{3} - \frac{x^3}{6}\Big|_1^2$$

$$\left[4 + \frac{32}{10} - \frac{8}{3} - \frac{8}{6}\right] - \left[\frac{1}{4} + \frac{1}{10} - \frac{1}{3} - \frac{1}{6}\right]$$

$$3.2 + 0.15 = \boxed{3.35}$$

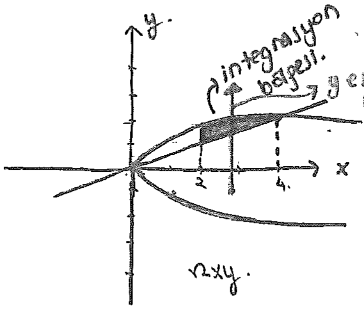
örnek: $\iint_R xy\, dA$ $\quad R:\ y=\frac{x}{2}\ ,\ y=\sqrt{x}\ ,\ x=2$ ve $x=4$.

$y^2 = x \qquad \frac{x}{2}=\sqrt{x}$

$\frac{x^2}{4} = x \rightarrow x^2-4x=0$

$x(x-4)=0$

$x=0 \quad x=4.$



integrasyon bölgesi. y eksenine paralel bir doğru çizerek x ve y'nin nerden nereye olduğunu belirliyoruz. Aynı zamanda y eksenine paralel bir doğru çizdiğimiz için integralde önce dy'yi yazıyoruz.

$$\int_2^4\int_{x/2}^{\sqrt{x}} xy\,dy\,dx$$

$$= \int_2^4 \frac{xy^2}{2}\Big|_{x/2}^{\sqrt{x}} \cdot dx$$

$$= \int_2^4\left(\frac{x^2}{2}-\frac{x^3}{8}\right)\cdot dx = \left[\frac{x^3}{6}-\frac{x^4}{32}\right]\Big|_2^4$$

$I = \left(\frac{64}{6}-8\right)-\left(\frac{8}{6}-\frac{1}{2}\right)$

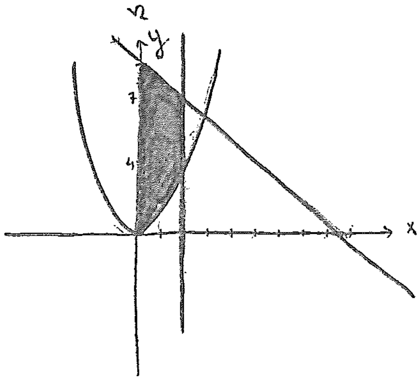
$I = \frac{56}{6}-\frac{15}{2} \rightarrow I = \frac{56-45}{6} \quad \boxed{I=\frac{11}{6}}$

örnek: R bölgesi; $0\le x\le 2 \quad x^2\le y\le 9-x$ olduğuna göre R bölgesi üzerinde;

$\iint_R (2y-x^2)\cdot dA = ?$

$y=x^2$

$y=9-x$



$$\iint_{Rxy}(2y-x^2)\,dy\,dx = \int_0^2\int_{x^2}^{9-x}(2y-x^2)\cdot dy\,dx$$

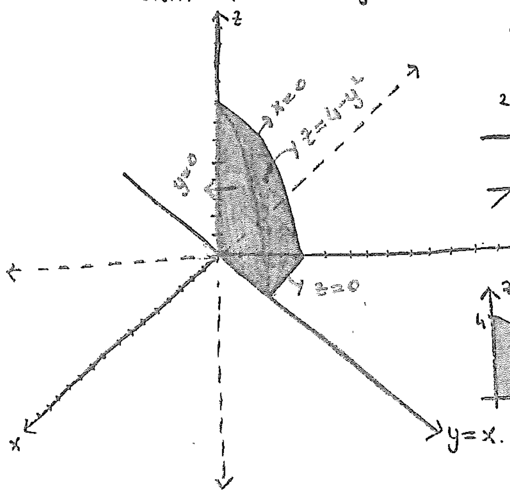
$$= \int_0^2\left(y^2-x^2y\right)\Big|_{x^2}^{9-x}\cdot dx = \int_0^2\left[\left[(9-x)^2-x^2(9-x)\right]-\left[\cancel{x^4-x^4}\right]\right]\cdot dx$$

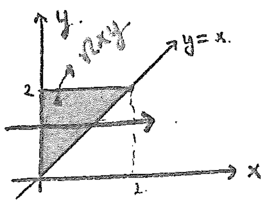
$$= \int_0^2\left[(9-x)^2-9x^2+x^3\right]dx = \left[-\frac{1}{3}(9-x)^3-3x^3+x^4/4\right]\Big|_0^2$$

$$= -\frac{1}{3}7^3-24+4+\frac{1}{3}9^3 = -\frac{1}{3}\cdot 343-20+\frac{1}{3}631$$

$$= +\frac{288}{3}-20 = \frac{+288-60}{3} = \boxed{\frac{228}{3}}$$

örnek: ilk 1/8'lik kısımda xy- düzlemi, yz- düzlemi, $y=x$ düzlemi ve $z=4-y^2$ parabolik silindiri ile bölgenin hacmini hesaplayınız





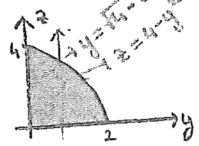
$$V = \iint_{Rxy}(4-y^2)\cdot dA = \int_0^2\int_0^y(4-y^2)\cdot dx\,dy$$

$$= \int_0^2(4x-y^2x)\Big|_0^y\cdot dy = \int_0^2(4y-y^3)\,dy.$$

$$= \left(2y^2-y^4/4\right)\Big|_0^2$$

$$= (8-4)$$

$\boxed{V=4}$

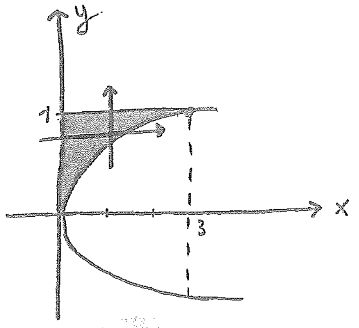


$$V = \iint_{Ryz} y\cdot dA = \int_0^2\int_0^{4-y^2} y\cdot dz\,dy.$$

↳ R bölgesini bu şekilde de seçebiliriz.

Örnek: Aşağıdaki integralleri integrasyon sırasını değiştirerek hesaplayınız.

a) $I_1 = \int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3} \cdot dy \cdot dx$ $\qquad y=1 \qquad y=\sqrt{\frac{x}{3}} \qquad y^2=\frac{x}{3}$



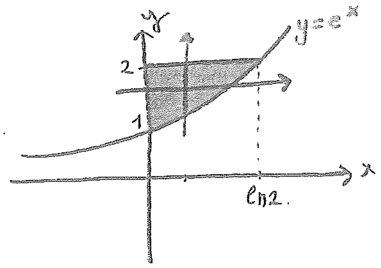
$$I_1 = \int_0^1 \int_0^{3y^2} e^{y^3} \cdot dx \cdot dy$$

$$= \int_0^1 e^{y^3} \cdot x \Big|_0^{3y^2} \cdot dy$$

$$= \int_0^1 e^{y^3} \cdot 3y^2 \cdot dy$$

$$= e^{y^3} \Big|_0^1 \rightarrow \boxed{V = e^1 - 1}$$

b) $I_2 = \int_1^2 \int_0^{\ln y} e^{-x} \cdot dx \cdot dy$ $\qquad x=0 \qquad x=\ln y \qquad y=e^x$



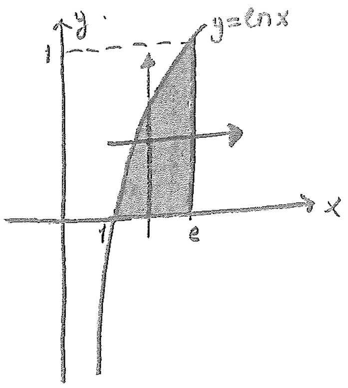
$$I_2 = \int_0^{\ln 2} \int_{e^x}^2 e^{-x} \cdot dy \cdot dx$$

$$= \int_0^{\ln 2} e^{-x} \cdot y \Big|_{e^x}^2 \cdot dx$$

$$= \int_0^{\ln 2} (2e^{-x} - 1) \cdot dx = (-2e^{-x} - x) \Big|_0^{\ln 2}$$

$$= -2 \cdot \frac{1}{2} - \ln 2 + 2 \rightarrow \boxed{I_2 = 1 - \ln 2}$$

c) $I_3 = \int_1^e \int_0^{\ln x} y \, dy \cdot dx$ $\qquad y=0 \qquad y=\ln x \qquad x=e^y$



$$I_3 = \int_0^1 \int_{e^y}^{e} y \cdot dx \cdot dy$$

$$= \int_0^1 yx \Big|_{e^y}^{e} dy$$

$$= \int_0^1 (e \cdot y - e^y \cdot y) \, dy = \int_0^1 y(e - e^y) \, dy$$

$$= \int_0^1 ey \, dy - \underbrace{\int_0^1 y \cdot e^y \cdot dy}_{I}$$

kısmi integrasyon!

$y=u \qquad e^y \cdot dy = dv$

$dy = du \qquad e^y = v$

$$= \frac{ey^2}{2} \Big|_0^1 - (e^y \cdot y - e^y) \Big|_0^1$$

$I = e^y \cdot y - \int e^y \cdot dy$

$\boxed{I = e^y \cdot y - e^y}$

$$= \frac{e}{2} - (e - e - (-1))$$

$$\boxed{I_3 = \frac{e}{2} - 1}$$

Örnek: $y=x$, $x=4y-y^2$ eğrilerinin sınırladığı bölgenin alanını iki katlı integralle hesaplayınız.

$$y^2-4y=-x$$

$$(y-2)^2+x=4$$

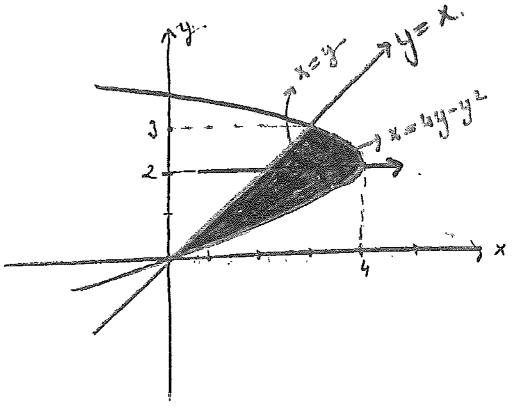
$$(y-2)^2=4-x$$

$$y=4y-y^2$$

$$y^2-3y=0$$

$$y(y-3)=0$$

$y=0$ $y=3$



$$A=\iint_{R_{xy}} dA = \int_0^3\int_y^{4y-y^2} dx\,dy$$

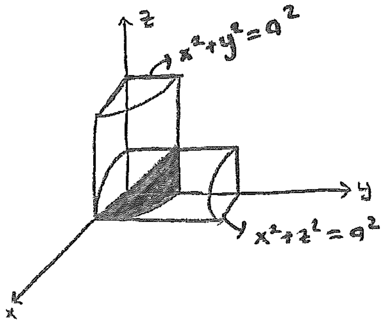
$$A=\int_0^3 x\Big|_y^{4y-y^2} . dy$$

$$=\int_0^3 (4y-y^2-y).dy$$

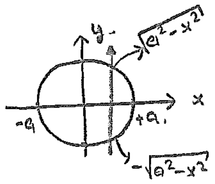
$$=-\int_0^3 (y^2-3y).dy = -\left(y^3/3 - 3y^2/2\right)\Big|_0^3$$

$$=-\left(9-27/2\right) \Rightarrow \boxed{A=\frac{9}{2}}$$

Örnek: $x^2+y^2=a^2$ ve $x^2+z^2=a^2$ silindirlerinin her ikisinin içinde kalan bölgenin hacmini iki katlı integralle hesaplayınız.



I. yol:



$$V=2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,.\,dy\,.\,dx.$$

$$V=2\int_{-a}^{a}[(a^2-x^2)+(a^2-x^2)]dx.$$

$$V=2\int_{-a}^{a}(2a^2-2x^2).\,dx$$

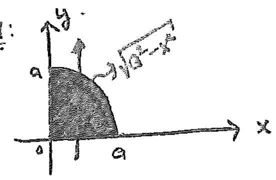
$$V=4\int_{-a}^{a}(a^2-x^2).\,dx$$

$$V=4\left[(a^2x-x^3/3)\right]_{-a}^{a}.\,dx.$$

$$V=4\left(a^3-\frac{a^3}{3}+a^3-\frac{a^3}{3}\right)$$

$$\boxed{V=\frac{16a^3}{3}}$$

II. yol:



$$V=8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}.dy.dx$$

$$V=8\int_0^a(a^2-x^2).\,dx$$

$$V=8\left[a^2x-x^3/3\right]\Big|_0^a$$

$$V=8(a^3-a^3/3)$$

$$\boxed{V=\frac{16a^3}{3}}$$

İki Katlı İntegrallerde Simetri

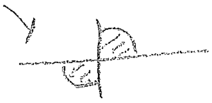
$$\int_{-a}^{a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx \rightarrow f \text{ çift ise ...}$$

NOT: İki katlı integralde simetrizlik için bölge simetrikliğine bakılır. x'e göre mi yoksa y'ye göre mi simetrik?

$$\int_{-a}^{a} f(x)\cdot dx = 0 \rightarrow f \text{ tek ise ...}$$

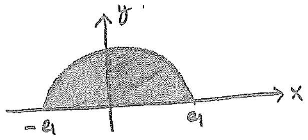
1) Ω Bölgesi y- eksenine göre simetrik olsun:

a) f fonksiyonu x'e göre tek ise $\{f(-x,y) = -f(x,y)\}$, $\iint_{\Omega} f(x,y)\cdot dA = 0$



b) f fonksiyonu x'e göre çift ise $\{f(-x,y) = f(x,y)\}$

$$\iint_{\Omega} f(x,y)\, dA = 2\iint_{\Omega\text{'nın sağ yarısı}} f(x,y)\cdot dA.$$

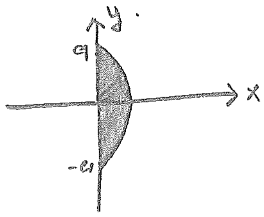


2) Ω bölgesi x- eksenine göre simetrik olsun:

a) f fonksiyonu y'ye göre tek ise $\{f(x,-y) = -f(x,y)\}$, $\iint_{\Omega} f(x,y)\cdot dA = 0$

b) f fonksiyonu y'ye göre çift ise $\{f(x,-y) = f(x,y)\}$,

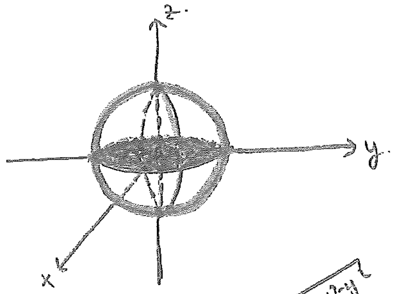
$$\iint_{\Omega} f(x,y)\, dA = 2\iint_{\Omega\text{'nın üst yarısı}} f(x,y)\cdot dA.$$

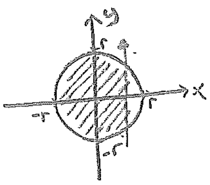


Örnek: merkezi orijinde olan r yarıçaplı kürenin hacmini hesaplayınız.

$$x^2+y^2+z^2 = r^2 \rightarrow z = \pm\sqrt{r^2-x^2-y^2}$$

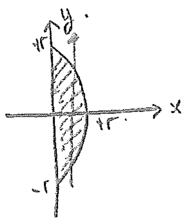
$$R_{xy} = x^2+y^2 \leq r^2.$$





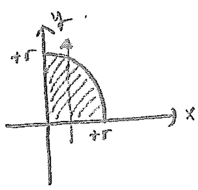
$$V = \int_{-r}^{+r}\int_{-\sqrt{r^2-x^2-y^2}}^{+\sqrt{r^2-x^2-y^2}} \sqrt{r^2-x^2-y^2}\cdot dy\, dx.$$

$$\boxed{V = 2\iint_{R_{xy}} \sqrt{r^2-x^2-y^2}\cdot dA}$$



$$V = 4\int_{0}^{r}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} \sqrt{r^2-x^2-y^2}\cdot dy\cdot dx.$$

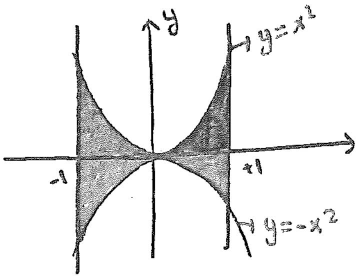
$$\boxed{V = 4\iint_{R_{xy}\text{'nin sağ yarısı}} \sqrt{r^2-x^2-y^2}\cdot dA}$$



$$V = \int_{0}^{r}\int_{0}^{\sqrt{r^2-x^2-y^2}} \sqrt{r^2-x^2-y^2}\cdot dy\cdot dx$$

$$\boxed{V = \iint_{R_{xy}\text{'nin sağ yarısının üst yarısı}} \sqrt{r^2-x^2-y^2}\cdot dA}$$

Örnek: $\iint_{R_{xy}} (x^4-2y)\,dA = \iint_{R_{xy}} x^4 \cdot dA - 2\iint_{R_{xy}} y \cdot dA$



$f_1(x,y) = x^4$ → y'ye göre çift fonksiyon

$\cancel{f_2(x,y) = y}$ → 0.

y'ye göre tek fonksiyon

$$4\cdot\int_0^1\int_0^{x^2} x^4 \cdot dy \cdot dx - 4\int_0^1\int_{-x^2}^{x^2} y\,dy\,dx$$

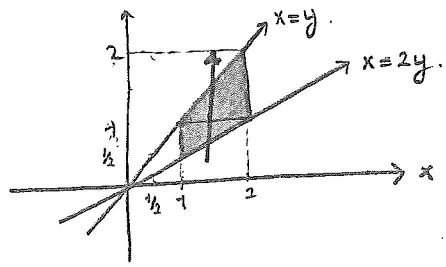
$$= 4\int_0^1 x^4 \cdot y\Big|_0^{x^2} \cdot dx - 4\int_0^1 \frac{y^2}{2}\Big|_{-x^2}^{x^2} dx$$

$$= 4\int_0^1 x^6 \cdot dx - 4\int_0^1 \left(\frac{x^4}{2} - \frac{x^4}{2}\right) \cdot dx.$$

$$= 4\left(\frac{x^7}{7}\Big|_0^1\right) - 4\int_0^1 0 \cdot dx = 4 \cdot \frac{1}{7} - 0 \quad \boxed{I = \frac{4}{7}}$$

Örnek: Aşağıdaki integralleri integrasyon sırasını değiştirerek veya iki katlı tek integrale dönüştürerek hesaplayınız.

a) $I_1 = \int_{1/2}^{1}\int_{1}^{2y} \frac{\ln x}{x}\,dx\,dy + \int_1^2\int_y^2 \frac{\ln x}{x} \cdot dx\,dy$



$x = 1$ $\quad x = y$

$x = 2y.$ $\quad x = 2.$

$$I_1 = \int_1^2\int_{x/2}^{x} \frac{\ln x}{x} \cdot dy \cdot dx$$

$$= \int_1^2 \frac{\ln x}{x} \cdot y\Big|_{x/2}^{x} \cdot dx = \int_1^2 \left(\ln x - \frac{\ln x}{2}\right) \cdot dx$$

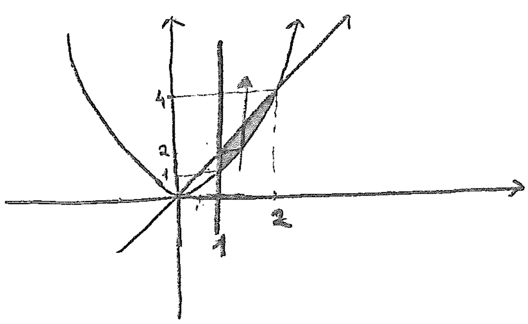
$$= \Big[(x.\ln x - x) - \tfrac{1}{2}(x.\ln x - x)\Big]_1^2 = \frac{1}{2}(x.\ln x - x)\Big|_1^2$$

$$\boxed{I = \ln 2 - \frac{1}{2}}$$

b) $I_2 = \int_1^2\int_1^{\sqrt{y}} dx\,dy + \int_2^4\int_{y/2}^{\sqrt{y}} dx\,dy. = \int_1^2\int_{x^2}^{2x} dy\,dx.$

$x = 1$ $\quad x = y/2$ $\quad y = x^2.$

$x = \sqrt{y}$ $\quad x = \sqrt{y}$ $\quad y = 2x.$



$$= \int_1^2 y.\Big|_{x^2}^{2x} \cdot dx$$

$$= \int_1^2 (2x - x^2)\,dx$$

$$= \left(x^2 - x^3/3\right)\Big|_1^2 dx$$

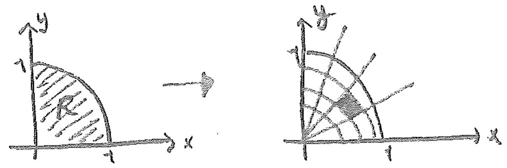
$$= \left(4 - \frac{8}{3}\right) - \left(1 - 1/3\right) \to \boxed{I_2 = \frac{2}{3}}$$

Kutupsal Koordinatlarla İki Katlı İntegralin Hesabı

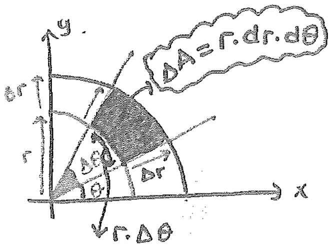
$$\iint_R (1-x^2-y^2)\cdot dA = ?$$ (Bu soruyu polar koordinatlarda çözmeye çalışalım.)

$x^2+y^2 \leq 1$

$(x, y \geq 0)$



* Elimizdeki integrasyon bölgesini belli açılarla ve daha küçük çember dilimleriyle keselim. ve oluşan küçük bölgenin alanını (ΔA) inceleyelim.



Bir çember diliminin uzunluğu $r.\Delta\theta$ ile bulunur.

* İncelemelerimiz sonucunda küçük bir çember dilimi için $dx\,dy$ ifadesini yani dA'yı $r.dr.d\theta$ şeklinde yazabiliyoruz.

$$I = \iint_R f.r.dr.d\theta$$ → Şimdi ise f fonksiyonu için r, θ cinsinden bir ifade bulalım.

$f = 1-x^2-y^2$ $\quad x = r.\cos\theta$

$\quad y = r.\sin\theta$

$f = 1-r^2\cos^2\theta - r^2\sin^2\theta$

$f = 1-r^2(\cos^2\theta + \sin^2\theta)$

$f = 1-r^2$

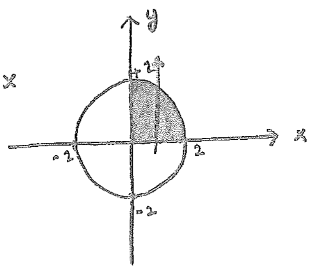
$$\boxed{I = \int_0^{\pi/2}\int_0^1 (1-r^2).r.dr.d\theta}$$

$$I = \int_0^{\pi/2}\left(r^2/2 - r^4/4\right)\Big|_0^1 d\theta = \int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{4}\right)d\theta = \int_0^{\pi/2} 1/4 . d\theta = \left[\frac{1}{4}\theta\right]_0^{\pi/2}$$

$$\boxed{I = \frac{\pi}{8}}$$

Örnek: Aşağıdaki integralleri kutupsal koordinatlar ile hesaplayınız.

a) $$I_1 = \int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,.dy.dx$$



$$I_1 = \int_0^{\pi/2}\int_0^2 r^2.dr.d\theta$$

$$= \int_0^{\pi/2} r^3/3 \Big|_0^2 d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}.d\theta$$

$$= \left[\frac{8}{3}\theta\right]_0^{\pi/2}$$

$$\boxed{I_1 = \frac{4\pi}{3}}$$

$y = 0$

$y = \sqrt{4-x^2}$

$x^2+y^2 = 4.$

$x = r.\cos\theta$

$y = r.\sin\theta$

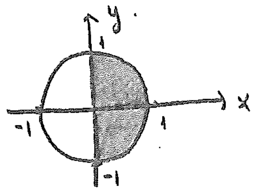
$x^2+y^2 = r^2$

b) $I_2 = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\,.dx\,dy.$

$x = 0$

$x = \sqrt{1-y^2}$

$x^2+y^2=1$



$I_2 = \int_{-\frac{\pi}{2}}^{\pi/2}\int_{0}^{1} r^2 . dr\, d\theta$

$= \ldots$

c) $I_3 = \int_{0}^{3/2}\int_{\sqrt{3}x}^{\sqrt{9-x^2}} e^{-x^2-y^2} . dy . dx.$

$y = \sqrt{3}\,x$

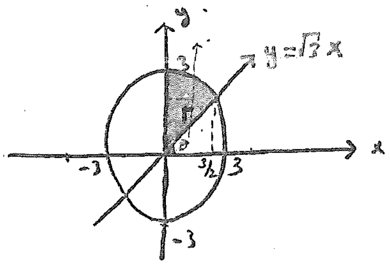
$y = \sqrt{9-x^2}$

$x^2+y^2=9.$

$3x^2 = 9 - x^2$

$4x^2 = 9$

$x = \pm\frac{3}{2}$



$\sin\theta = \frac{\sqrt{3}.\cancel{3}}{2.\cancel{3}}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$

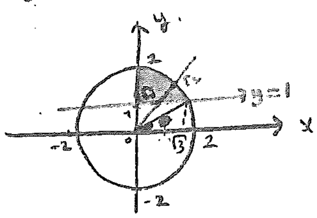
$I_3 = \int_{\frac{\pi}{3}}^{\pi/2}\int_{0}^{3} e^{-r^2} . r . dr . d\theta$

d) $I_4 = \int_{0}^{\sqrt{3}}\int_{1}^{\sqrt{4-x^2}} \frac{y}{x^2+y^2} . dy\, dx$

$y = 1$

$y = \sqrt{4-x^2}$

$x^2+y^2=4.$



$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$y = r_1 . \sin\theta$

$1 = r_1 . \sin\theta$

$r_1 = \frac{1}{\sin\theta}$

$r_2 = 2$

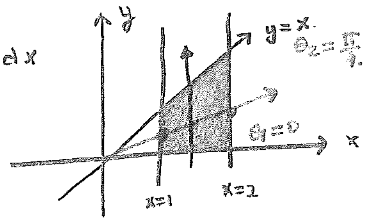
$I_4 = \int_{\frac{\pi}{6}}^{\pi/2}\int_{\frac{1}{\sin\theta}}^{2} \sin\theta . dr\, d\theta .$

Örnek: Aşağıdaki integralleri kutupsal koordinatlar yardımıyla çözünüz

a) $I_1 = \int_{1}^{2}\int_{0}^{x} 1/\sqrt{x^2+y^2} . dy . dx$

$y = 0$

$y = x$



$x = r_1 . \cos\theta$

$1 = r_1 . \cos\theta$

$r_1 = \sec\theta$

$x = r_2 . \cos\theta$

$2 = r_2 . \cos\theta$

$r_2 = 2 . \sec\theta$

$I_1 = \int_{0}^{\pi/4}\int_{\sec\theta}^{2\sec\theta} dr\, d\theta .$

b) $I_2 = \int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} \sin(x^2+y^2)\, dx\, dy$

$I_2 = \int_0^{\pi/4} \int_0^1 \sin r^2 \cdot r\, dr\, d\theta$

$\vdots$

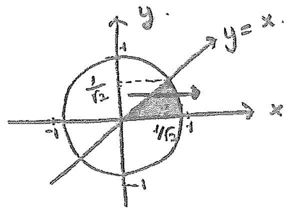
$x = y$

$x = \sqrt{1-y^2}$

$x^2+y^2 = 1$

$2x^2 = 1$

$x = \mp \frac{1}{\sqrt{2}}$



$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

c) $I_3 = \int_0^{\frac{\sqrt{27}}{2}} \int_{\frac{x}{\sqrt{3}}}^{\sqrt{9-x^2}} \frac{1}{\sqrt{x^2+y^2}} \cdot dy\, dx$

$\int_{\frac{\pi}{6}}^{\pi/2} \int_0^3 dr\, d\theta$

$\vdots$

$y = \frac{x}{\sqrt{3}}$

$y = \sqrt{9-x^2}$

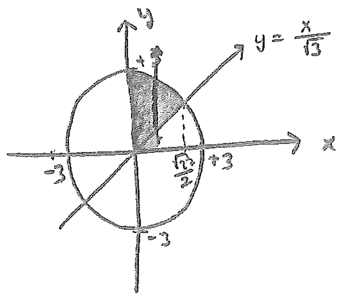
$x^2+y^2 = 9$

$x^2 + \frac{x^2}{3} = 9$

$\frac{4x^2}{3} = 9$

$x^2 = \frac{27}{4}$

$x = \frac{\sqrt{27}}{2}$



$\cos\theta = \frac{\sqrt{27}}{2 \cdot 3}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$

d) $\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2)\, dy\, dx$

$I_4 = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^3 \cdot dr\, d\theta$

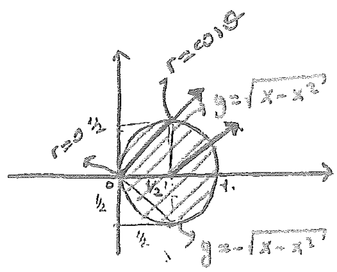
$y = -\sqrt{x-x^2}$ $\quad$ $y = \sqrt{x-x^2}$

$x^2+y^2 = x$ $\quad$ $y^2 = x - x^2$

$r^2 = r \cdot \cos\theta$ $\quad$ $x^2 - x + y^2 = 0$

$r(r - \cos\theta) = 0$ $\quad$ $(x - 1/2)^2 + y^2 = \frac{1}{4}$

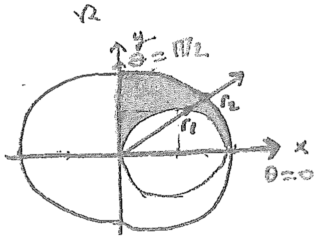
$\boxed{r_1 = 0 \quad r_2 = \cos\theta}$



$\boxed{\begin{array}{l} \sin\theta_1 = -\pi/2 \\ \sin\theta_2 = +\pi/2 \end{array}}$

örnek: $\iint_{\Omega} dA = ?$ Ω: 1. bölgede $x^2+y^2=4$ ve $x^2+y^2=2x$ 'in arasında kalan bölge;

$(x-1)^2+y^2 = 1$



$x^2+y^2 = 2x$

$r^2 = 2 \cdot r\cos\theta$

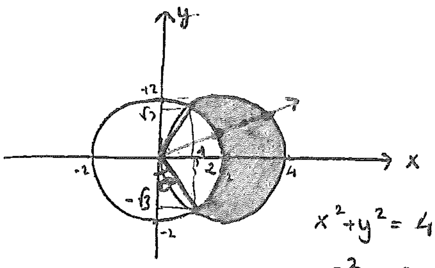
$r(r - 2\cos\theta) = 0$

$\boxed{r = 0 \quad r = 2\cos\theta}$

$A = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot dr\, d\theta$

$\vdots$

Örnek: $(x-2)^2+y^2=4$ çemberinin içinde $x^2+y^2=4$ çemberinin dışında kalan bölgenin alanını iki katlı integral ile hesaplayınız. Ayrıca integrasyon bölgesini çiziniz.



$$A=\iint_\Omega dA = \int_{-\pi/6}^{\pi/6}\int_{2}^{4\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{r^2}{2}\Big|_{2}^{4\cos\theta} d\theta$$

$x^2+y^2=4x$

$r^2=4r\cos\theta$

$r(r-4\cos\theta)=0$

$r=0 \quad r=4\cos\theta$

$$= \int_{-\frac{\pi}{6}}^{\pi/6}\left(\frac{16\cos^2\theta}{2}-\frac{4}{2}\right)d\theta$$

$x^2+y^2-4x+4=4$

$x^2+y^2=4x$

$4x=4$

$\boxed{x=1}$

$\cos\theta_1=-\frac{\sqrt{3}}{2}$

$\theta_1=\frac{-\pi}{6}=-30^\circ$

$\sin\theta_2=\frac{\sqrt{3}}{2}$

$\theta_2=\frac{\pi}{3}$

$\boxed{\begin{array}{l}\theta_1=-\frac{\pi}{6}\\ \theta_2=\frac{\pi}{6}\end{array}}$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(8\cos^2\theta-2)\,d\theta.$$

$2\cos^2\theta-1=\cos2\theta$

$\cos^2\theta=\frac{\cos2\theta+1}{2}.$

$$4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(\cos2\theta+1)\,d\theta-2\int_{-\frac{\pi}{6}}^{\pi/6}d\theta = 2\left(\sin2\theta\Big|_{-\frac{\pi}{6}}^{\pi/6}\right)-2(\theta)_{-\frac{\pi}{6}}^{\pi/6}$$

$$=2\cdot\left(\sin\frac{\pi}{3}-\sin\left(-\frac{\pi}{3}\right)\right)-2\left(\frac{\pi}{6}+\frac{\pi}{6}\right)+4\left(\frac{\pi}{6}+\frac{\pi}{6}\right)$$

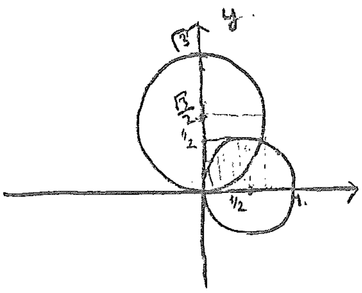
$$=2\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)+$$

Örnek: $x^2+y^2=x$ ve $x^2+y^2=\sqrt{3}\,y$ eğrilerinin sınırladığı bölgenin alanını iki katlı integralle hesaplayınız.

$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$x^2+(y-\frac{\sqrt{3}}{2})^2=\frac{3}{4}.$

$$A=\iint_\Omega dA = \iint r\cdot dr\cdot d\theta.$$



$x^2+y^2=x$

$r^2=r\cdot\cos\theta$

$r(r-\cos\theta)=0$

$r=0 \quad r=\cos\theta.$

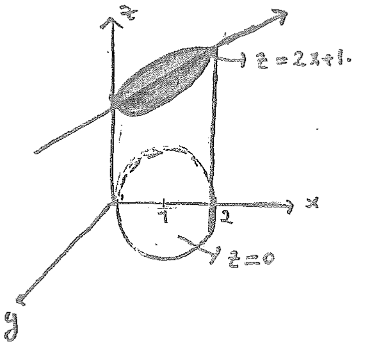
$x^2+y^2=\sqrt{3}\,y$

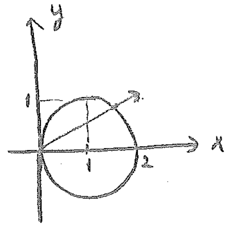
$r^2=\sqrt{3}\cdot r\cdot\cos\theta$

$r(r-\sqrt{3}\cos\theta)=0$

$r=0 \quad r=\sqrt{3}\cos\theta.$

Örnek: Üstten $z=2x+1$ düzlemi alttan $(x-1)^2+y^2\le 1$ ile sınırlı bölgenin hacmini iki katlı integralle hesaplayınız.





$$V=\iint_{\Omega_{xy}}(2x+1)\,dA$$

$$\int_0^2\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}(2x+1)\,dy\,dx$$

$$x^2-2x+1+y^2=1$$
$$y=\sqrt{2x-x^2}$$

$$x^2+y^2=2x$$
$$r^2=2.r\cos\theta$$
$$r(r-2\cos\theta)=0$$
$$r=0 \quad r=2\cos\theta$$

$$=\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta}(2r\cos\theta+1)\,r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}(2r^2\cos\theta+r)\,dr\,d\theta.$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{2}{3}r^3\cos\theta+\frac{r^2}{2}\right)\Big|_0^{2\cos\theta}\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{16}{3}\cos^4\theta+2\cos^2\theta\right)d\theta.$$

$$\frac{16}{3}\int(1-\sin^2\theta)(1-\sin^2\theta)\,d\theta.\ 2\int(1-\sin^2\theta)\,d\theta$$

$$\frac{16}{3}\int\left(\frac{1-\cos2\theta}{2}\right)\left(\frac{1-\cos2\theta}{2}\right)d\theta+2\int\left(1-\frac{1-\cos2\theta}{2}\right)d\theta$$

Örnek: $I=\int_0^\infty e^{-x^2}.dx=?$

$$I^2=\int_0^\infty\int_0^\infty e^{-x^2}.e^{-y^2}.dy\,dx$$

$$I^2=\int_0^{\pi/2}\int_0^\infty e^{-r^2}.r.dr.d\theta.$$

$$\int_0^\infty e^{-r^2}.r.dr=\lim_{b\to\infty}\int_0^b e^{-r^2}.r.dr$$

$$=\lim_{b\to\infty}\left(-\frac{1}{2}e^{-r^2}\Big|_0^b\right)$$

$$=-\frac{1}{2}\lim_{b\to\infty}\left[\cancel{e^{-b^2}}^{\,0}+1\right]$$

$$I^2=\frac{1}{2}\int_0^{\pi/2}d\theta=\frac{\pi}{4}\rightarrow\boxed{I=\frac{\sqrt{\pi}}{2}}$$

Değişken Değişimi Yardımıyla İki Katlı İntegral Çözümü

$$\iint_R f(x,y)\,dx\,dy = \iint_P f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\cdot du\cdot dv.$$

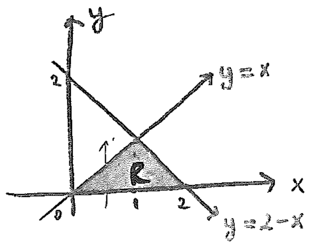
$x = r\cdot\cos\theta$
$y = r\cdot\sin\theta$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_P f(u,v)\cdot r\cdot du\cdot dv}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$$

Örnek: $\iint_R xy\,dA$ $\quad x = u+v$, $y = u-v$; $R: y=x$, $y=2-x$ ve $y=0$

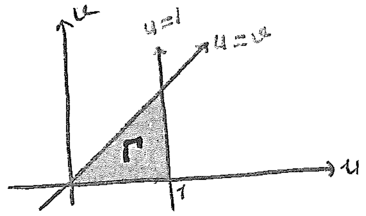


$y = x \rightarrow u+v = u-v \rightarrow v = 0$

$y = 2-x \rightarrow u-v = 2-u-v \rightarrow u = 1$

$y = 0 \rightarrow u-v = 0 \rightarrow u = v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$



$$\iint_R f(x,y)\,dA = \iint_\Gamma (u^2-v^2)\cdot|-2|\cdot du\cdot dv$$

$$2\int_0^1\int_0^u (u^2-v^2)\cdot dv\cdot du = 2\int_0^1 \left(u^2v - \frac{v^3}{3}\right)\Big|_0^u du$$

$$= 2\int_0^1 \left(u^3 - \frac{u^3}{3}\right) du = \frac{4}{3}\int_0^1 u^3\,du = \frac{4}{3}\left(\frac{u^4}{4}\Big|_0^1\right)$$

$$= \frac{4}{3}\left(\frac{1}{4}\right) \rightarrow \boxed{A = \frac{1}{3}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$x+y = 2u$
$x-y = 2v$
$u = \frac{x+y}{2}$
$v = \frac{x-y}{2}$

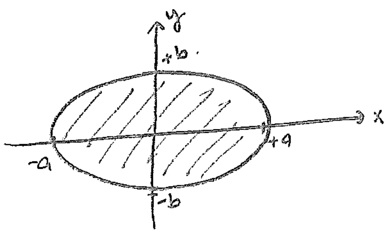
$$= \frac{1}{\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}}$$

$$= \frac{1}{-\frac{1}{2}} = -2$$

Örnek: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ elipsinin alanını hesaplayınız.

$\frac{x}{a} = u$, $\frac{y}{b} = v \rightarrow u^2+v^2 = 1$

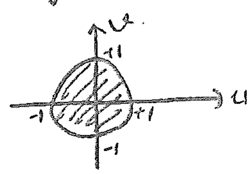
$u = r\cos\theta$
$v = r\cdot\sin\theta$
$x = a\cdot r\cos\theta$
$y = b\cdot r\cdot\sin\theta$.



$$\iint_R dA = \iint_\Gamma \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\cdot dr\cdot d\theta$$

$$= \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta$$

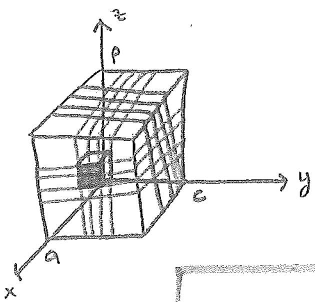
$$= ab\pi$$



$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

Üç katlı integraller

$R: \{ a \le x \le b \ , \ c \le y \le d \ , \ p \le z \le q \}$



$\left.\begin{array}{l} x = x_i \\ y = y_j \\ z = z_k \end{array}\right\}$ Düzlemler $\begin{array}{l}(i=1,\dots,m) \\ (j=1,\dots,n) \\ (k=1,\dots,r)\end{array}$ $\quad \begin{array}{l} x_{i-1} \le x \le x_i \\ y_{j-1} \le y \le y_j \\ z_{k-1} \le z \le z_k \end{array}$

$\Delta V_{ijk} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$ $\quad$ f, R bölgesinde tanımlanmış olsun.

$$\boxed{\lim_{\substack{m,n,r \to \infty \\ \max\{\Delta x_i, \Delta y_j, \Delta z_k\} \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} f(\bar{x}_i, \bar{y}_j, \bar{z}_k) \cdot \Delta V_{ijk}}$$

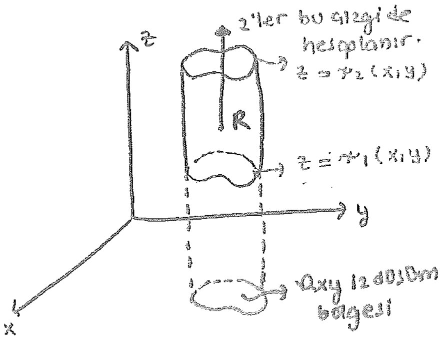
Yandaki limit mevcut ise bu limit değeri f fonksiyonunun R bölgesi üzerinde 3 katlı integrali olr. adlandırılır ve bu integral sembolik olarak;

$\boxed{\iiint_R f(x,y,z) \cdot dV}$ şeklinde gösterilir.

$dV = dx\,dy\,dz \rightarrow$ hacim elemanı

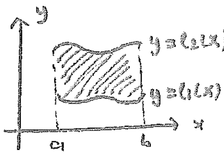
* R bölgesinin hacmi; $\iiint_R dV$ ile hesaplanır

Üç katlı integralin iteratif hesaplanması



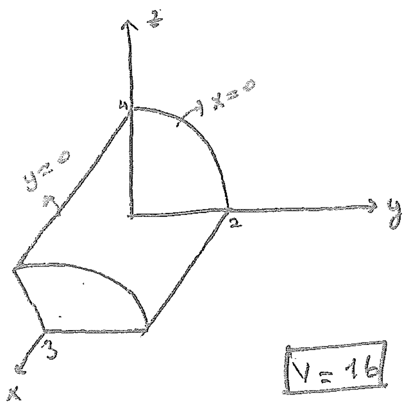
$$\iiint_R f(x,y,z) \cdot dV = \iint_{\Omega_{xy}} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z) \cdot dz \cdot dA$$

$\Omega_{xy} = \{ a \le x \le b \ , \ l_1(x) \le y \le l_2(x) \}$

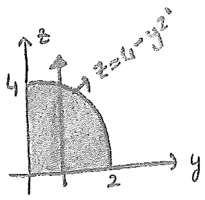


$$\boxed{\iiint_R f(x,y,z)\, dV = \int_a^b \int_{l_1(x)}^{l_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z) \cdot dz\, dy\, dx}$$

Örnek: İlk 1/8'lik kısımda koordinat düzlemleri, $x=3$ düzlemi ve $z = 4-y^2$ parabolik silindiri ile sınırlı bölgenin hacmini 3 katlı integral ile hesaplayınız.



yz- düzlemine izdüşürelim



$\boxed{V = 16}$

$$V = \int_0^2 \int_0^{4-y^2} \int_0^3 dx\, dz\, dy$$

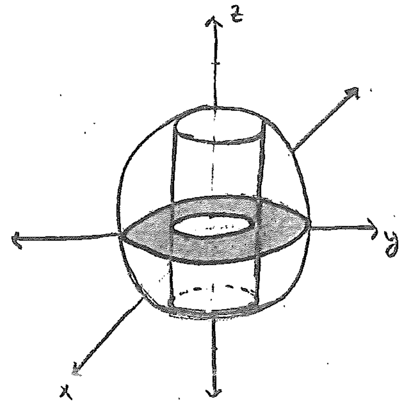
$$V = \int_0^2 \int_0^{4-y^2} x \Big|_0^3 \, dz\, dy$$

$$= 3 \int_0^2 \int_0^{4-y^2} dz\, dy = 3 \int_0^2 z \Big|_0^{4-y^2} \cdot dy$$

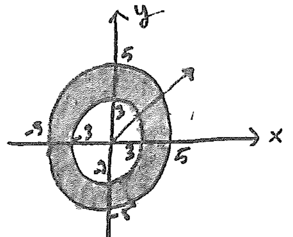
$$= 3 \int_0^2 (4-y^2) dy = 3 \left(4y - \frac{y^3}{3} \Big|_0^2 \right)$$

$$= 3 \left(8 - \frac{8}{3} \right) \Rightarrow$$

Örnek: $x^2+y^2+z^2=25$ küresinin içinde ve $x^2+y^2=9$ silindirinin dışında kalan bölgenin hacmini 3 katlı integralle hesaplayınız.



$$z=\pm\sqrt{25-x^2-y^2}$$



$$V=\iint_{R_{xy}}\int_{-\sqrt{25-x^2-y^2}}^{+\sqrt{25-x^2-y^2}} dz.dxdy$$

$$V=2\iint_{R_{xy}}\int_0^{\sqrt{25-x^2-y^2}} dz.dA$$

$x=r\cdot\cos\theta$

$y=r\cdot\sin\theta$

$z=z$

$$=2\iint_{R_{xy}}\sqrt{25-x^2-y^2}.dA$$

$$=2\int_0^{2\pi}\int_3^5\sqrt{25-r^2}\cdot r\,dr\,d\theta$$

$25-r^2=t^2$

$-2r\,dr=2t\,dt$

$r\,dr=-t.dt$

$r=3 \quad t=+4$

$r=5 \quad t=0$

$$=-2\int_0^{2\pi}\int_{+4}^{0} t^2.dt.d\theta$$

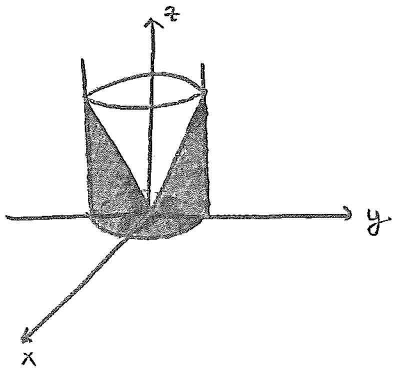
$$=-2\int_0^{2\pi}\left(\frac{t^3}{3}\Big|_{+4}^{0}\right)d\theta$$

$$=-2\int_0^{2\pi}\left(0-\frac{64}{3}\right)d\theta=\frac{128}{3}\left(\theta\Big|_0^{2\pi}\right)$$

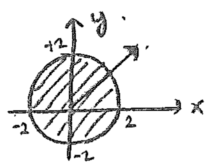
$$\boxed{V=\frac{256\pi}{3}}$$

Örnek: $z=\sqrt{x^2+y^2}$ yüzeyinin altında, $z=0$ düzleminin üstünde ve $x^2+y^2=4$ silindirinin içinde kalan bölgenin hacmini üç katlı integral ile hesaplayınız

$z^2=x^2+y^2$ (koni denklemi)



$$V=\iint_{R_{xy}}\int_0^{\sqrt{x^2+y^2}} dz.dA.$$



$$V=\iint_{R_{xy}}\sqrt{x^2+y^2}.dA$$

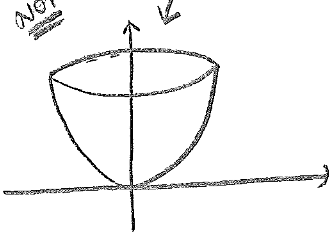
$$V=\int_0^{2\pi}\int_0^2 r^2dr\,d\theta$$

$$\int_0^{2\pi}\frac{r^3}{3}\Big|_0^2 d\theta.$$

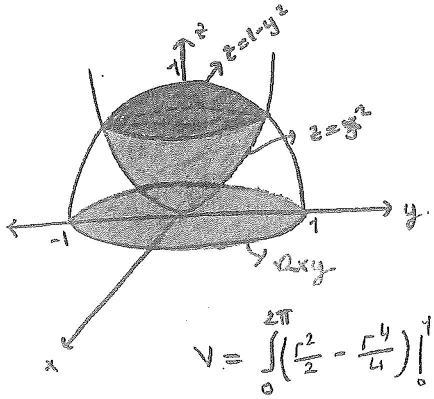
$$\int_0^{4\pi}\frac{8}{3}d\theta \rightarrow \frac{8}{3}\theta\Big|_0^{2\pi}$$

$$\boxed{V=\frac{16\pi}{3}}$$

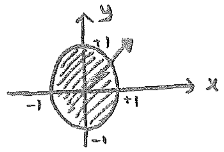
Not! $z=x^2+y^2$



Örnek: $z = 1-y^2$ yüzeyinin altında ve $z = x^2$ yüzeyinin üstünde kalan bölgenin hacmini üç katlı integral ile hesaplayınız.



$$\left.\begin{aligned} z &= 1-y^2 \\ z &= x^2 \end{aligned}\right\} \quad x^2+y^2=1$$



$$V = \iint_{R_{xy}} \int_{x^2}^{1-y^2} dz \, dA$$

$$V = \iint_{R_{xy}} (1-y^2-x^2) \cdot dA$$

$$V = \int_0^{2\pi}\int_0^{1} (1-r^2) r \, dr \, d\theta$$

$$V = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} d\theta = \frac{1}{4}\theta \Big|_0^{2\pi}$$

$$\boxed{V = \frac{\pi}{2}}$$

Silindirik koordinatlar

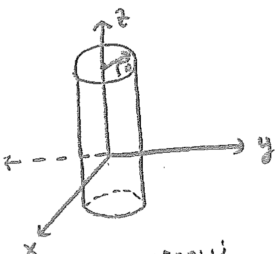
xyz uzayındaki bir P noktasının silindirik koordinatları (r, θ, z) ile gösterir.

Kartezyen koordinatlarda;

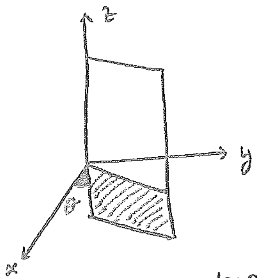
$x = x_0$, $y = y_0$, $z = z_0$ koordinat yüzeyleri birbirlerine dik düzlemlerdir.

Silindirik koordinatlarda bu yüzeyler;

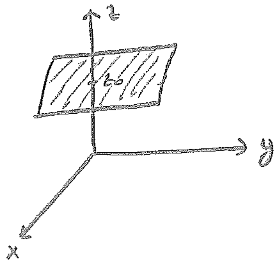
$r = r_0$, $\theta = \theta_0$, $z = z_0$ formundadırlar. Bu yüzeylerin şekilleri aşağıdaki gibidir.

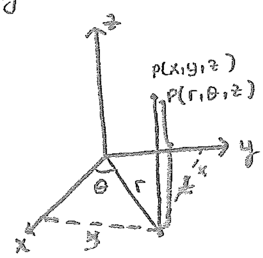


$r = r_0$ yüzeyi



z- eksenine dayalı yarı düzlem



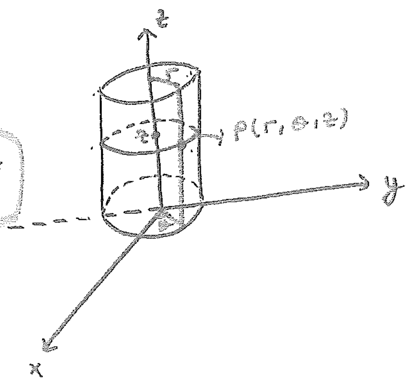


$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r.\sin\theta \\ z &= z \end{aligned}}$$

Her şey kutupsal koordinatlar ile aynı sadece bir z yüzeyi ekleniyor.

$$\iiint_R f(x,y,z) \cdot dA = \iiint_S F(r,\theta,z) \cdot \left|\frac{\partial(x,y,z)}{\partial(r,z,\theta)}\right| dr \, d\theta \, dz$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$



Örnek: Aşağıdaki integralleri silindirik koordinatlar yardımıyla hesaplayınız.

a) $I_1 = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} (x^2+y^2)\,dz\,dy\,dx$

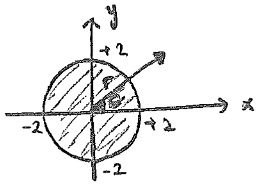
$x = r\cdot\cos\theta$

$y = r\cdot\sin\theta$

$z = z$

$y = \pm\sqrt{4-x^2}$

$x^2+y^2=4$



$I_1 = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2 \cdot dz\,.\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta$

$= \int_0^{2\pi} \left(r^4 - \frac{r^6}{6} \right)\Big|_0^2 d\theta = \int_0^{2\pi} \left(16 - \frac{64}{6}\right) d\theta$

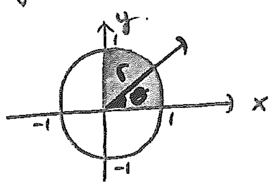
$V = \frac{32}{6}\int_0^{2\pi} d\theta = \frac{32}{6}\left(\theta\Big|_0^{2\pi}\right)$

$\boxed{V = \frac{32\pi}{3}}$

b) $I_2 = \int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 z^3 \,.\, dz\,dy\,dx$

$y^2 = 1-x^2$

$x^2+y^2=1.$



$I_2 = \int_0^{\pi/2}\int_0^1\int_r^1 z^3 \,.\, r\,dz\,dr\,d\theta$

$I_2 = \frac{1}{4}\int_0^{\pi/2}\int_0^1 r\,.\,z^4\Big|_r^1 \,dr\,d\theta$

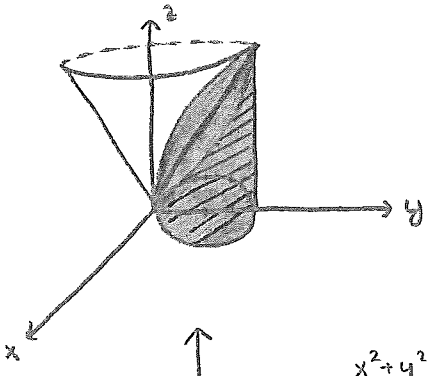
$= \frac{1}{4}\int_0^{\pi/2}\int_0^1 (r - r^5)\,dr\,d\theta$

$= \frac{1}{4}\int_0^{\pi/2}\left(\frac{r^2}{2} - \frac{r^6}{6}\right)\Big|_0^1 d\theta$

$I_2 = \frac{1}{4}\int_0^{\pi/2}\left(\frac{1}{3}\right)d\theta = \frac{1}{12}\left(\theta\Big|_0^{\pi/2}\right)$

$I_2 = \frac{\pi}{36}$

Örnek: Üstten $z^2 = x^2+y^2$ konisi alttan xy- düzlemi ve yandan $x^2+y^2=4x$ silindiri ile sınırlı bölgenin hacmini hesaplayınız.

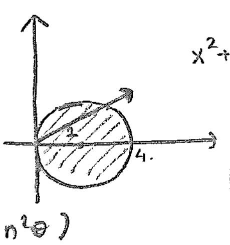


$z^2 = x^2+y^2$

$(x-2)^2+y^2=4.$

$V = \iint_{Rxy}\int_0^{\sqrt{x^2+y^2}} dz\,dy\,dx$

$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta}\int_0^r r\,dz\,dr\,d\theta.$



$x^2+y^2=4x$

$r^2 = 4r\cos\theta$

$r(r-4\cos\theta)=0$

$\boxed{r=0 \quad r=4\cos\theta}$

$= \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{4\cos\theta} r^2\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_0^{4\cos\theta} \cdot d\theta$

$\frac{64}{3}\int_{-\frac{\pi}{2}}^{\pi/2}\cos^3\theta\,d\theta = \frac{64}{3}\left(\sin\theta\Big|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} - \frac{1}{3}\sin^3\theta\Big|_{-\frac{\pi}{2}}^{\pi/2}\right)$

$\cos^3\theta = \cos\theta(1-\sin^2\theta)$

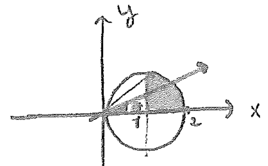
$\int\cos\theta\,d\theta - \int\cos\theta\,.\sin^2\theta\,d\theta -$

$\boxed{\sin\theta - \frac{1}{3}\sin^3\theta}$

$\boxed{V = \frac{256}{9}}$

Örnek: $I = \int_1^2 \int_0^{\sqrt{2x-x^2}} \int_0^{1/\sqrt{x^2+y^2}} dz\,dy\,dx$

$y = 0$
$y^2 = 2x - x^2$
$(x-1)^2 + y^2 = 1.$



$x^2+y^2 = 2x$
$r^2 = 2r\cos\theta$
$r(r-2\cos\theta) = 0$
$r=0 \quad r = \underline{2\cos\theta}$

$x = r\cdot\cos\theta$
$1 = r\cdot\cos\theta$
$\underline{r = \sec\theta}.$

$$I = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \int^{1/r} r\,dz\,dr\,d\theta.$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} dr\,d\theta = \int_0^{\pi/4} (2\cos\theta - \sec\theta)\,d\theta.$$

$$= 2\sin\theta \Big|_0^{\pi/2} - (\ln|\sec\theta + \tan\theta|)\Big|_0^{\pi/4}.$$

$$= 2 - \ln|\sec\pi/4 + 1| + \ln|1+0|$$

$$\boxed{I = 2 - \ln\sqrt{2}}$$

Küresel Koordinatlar

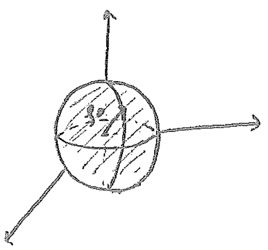
xyz- uzayında bir P noktasının küresel koordinatlarını (ρ, θ, ϕ) ile gösterelim. Bu koordinat sisteminde koordinat yüzeyleri;

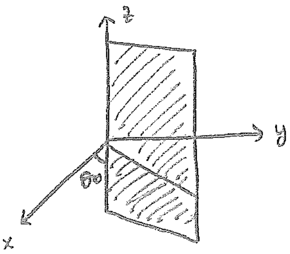
$\rho = \rho_0$; $\theta = \theta_0$; $\phi = \phi_0$ şeklindedir.

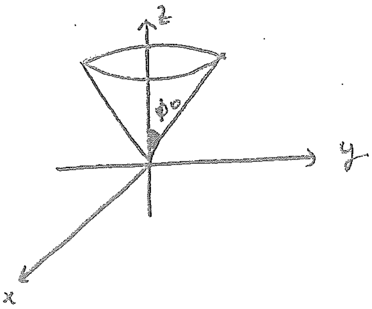
$\rho \geq 0$ (kürenin yarıçapı)

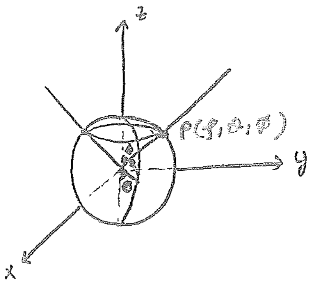
$\theta \in [0, 2\pi]$ (z- eksenine dayalı yarı düzlem)

$\phi \in [0, \pi]$ (koninin tepe açısı)







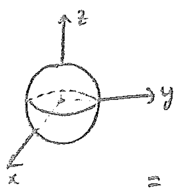


$$\iiint_R f(x,y,z)\,dV = \iiint F(\rho,\theta,\phi) \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| d\rho\, d\phi\, d\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi \qquad \underbrace{dx\,dy\,dz}_{dV} = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$x = r\cdot\cos\theta \rightarrow x = \rho\cdot\sin\phi\cos\theta$
$y = r\cdot\sin\theta \rightarrow y = \rho\,\sin\phi\cdot\sin\theta$
$z = \rho\cdot\cos\phi \rightarrow z = \rho\cdot\cos\phi$
$r = \rho\cdot\sin\phi \qquad x^2+y^2+z^2 = \rho^2$
$\tan\theta = \frac{y}{x} \rightarrow \theta = \arctan\left(\frac{y}{x}\right)$
$\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$

Örnek: Merkezi orijinde olan $x^2+y^2+z^2 = R^2$ küresinin hacmini hesaplayınız



$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3} \sin\phi\, d\phi\, d\theta.$$

$$= \frac{-R^3}{3}\int_0^{2\pi} \cos\phi \Big|_0^{\pi} \cdot d\theta = \frac{2R}{3}\left(\theta\Big|_0^{2\pi}\right) = \frac{4\pi R^3}{3}$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$

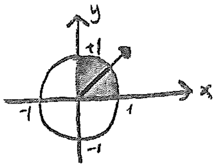
Örnek: Aşağıdaki integralleri küresel koordinatlar yardımıyla hesaplayınız

a) $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 1/x^2+y^2+z^2 \cdot dz\,dy\,dx$$

$y=0$

$y^2=1-x^2$

$x^2+y^2=1$



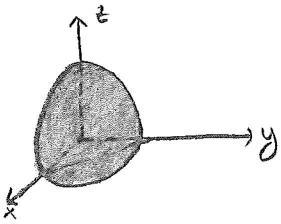
$z=0$

$z^2=1-x^2-y^2$

$x^2+y^2+z^2=1=\rho^2$

(Küre ama kürenin ilk 1/8'lik kısmı)

$\underline{\underline{\rho=1}}$



$r=\rho.\sin\phi$

$r=1\sin\phi$

$1=1.\sin\phi$

$\sin\phi=1 \quad \phi=\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi/2}\int_0^1 1/\rho^2\cdot\rho^2\cdot\sin\phi\,d\rho.d\phi\,d\theta$$

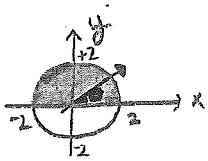
$$=\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta = +\int_0^{\pi/2}d\theta=\boxed{\frac{\pi}{2}}$$

b) $$I_2=\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$y=0$

$y^2=4-x^2$

$x^2+y^2=4$

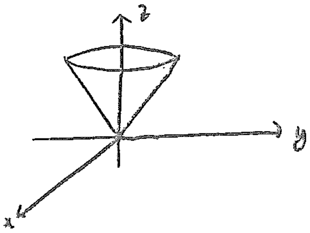


$x^2+y^2=z^2$ (koni)

$x^2+y^2+z^2=8$ (daire)

$8=\rho^2$

$\rho=2\sqrt{2}$



$r=\rho.\sin\phi$

$2=2\sqrt{2}.\sin\phi$

$\sin\phi=\frac{1}{\sqrt{2}} \quad \phi=\frac{\pi}{4}$

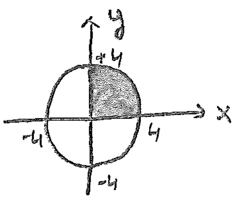
$$\boxed{I_2=\int_0^{\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}}\rho^3\sin\phi\,d\rho.d\phi\,d\theta}$$

c) $$I_3=\int_0^4\int_0^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4}\sqrt{x^2+y^2+z^2}.\,dz\,dy\,dx$$

$y=0$

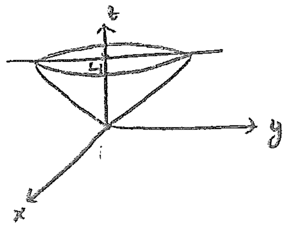
$y^2=16-x^2$

$x^2+y^2=16$



$z^2=x^2+y^2$ (koni)

$z=4.$



$r=\rho.\sin\phi \qquad z=\rho\cos\phi$

$4=\rho.\sin\phi \qquad 4=\rho.\cos\phi$

$1=\tan\phi\to\phi=\frac{\pi}{4}.$

$z=\rho.\cos\phi$

$\rho=\frac{4}{\cos\phi}$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{4/\cos\phi}\rho^3.\sin\phi.d\rho.d\phi\,d\theta.$$