System Theory, KOM5108
Lecture #3

Instructor: Dr. Yavuz Eren

Lecture Book: **"Linear System Theory, W.J. Rugh"**

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Orthogonality: Let $(\mathcal{X}, <.,.>)$ be an inner product space. Two vectors $x, y \in \mathcal{X}$ are said to be orthogonal, denoted by $x \perp y$ if $< x, y >= 0$.

A basis of $\mathcal{X}$ is orthogonal if $x_i \perp x_j$ for all i $\neq j$, where x_i's are basis vectors of $\mathcal{X}$. Moreover they are orthonormal bais if $< x, y >= 1$. In fact, given any basis of a subspace, one can always obtain an othonormal basis through Gram-Schmidt orthonormalization process as follows:

Lemma (Gram-Schmidt Orthonormalization): Let $(\mathcal{X}, <.,.>)$ be an inner product space and let $S \subseteq \mathcal{X}$ be a subspace of $\mathcal{X}$. Let $\{s_1, s_2, \ldots, s_r\}$ be a basis of S. Then, $\{s_1, s_2, \ldots, s_r\}$, where

$$\overline{s}_k = \begin{cases} \dfrac{s_1}{\|s_1\|} & for\ k = 1 \\ \dfrac{z_k}{\|z_k\|} & for\ k = 2 : r,\ where\ z_k := s_k - \displaystyle\sum_{i=1}^{k-1} < s_k, \overline{s}_i > \overline{s}_i \end{cases}$$

is a orthonormal basis of S.

Notes for implementation of G-SO:

Continued…

Product Operations(Inner, Outer and Cross)

Inner Products(Dot or Scalar Product): Inner products of the vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ is defined as:

$$< x, y >= x_1{}^* y_1 + x_2{}^* y_2 + \cdots + x_n{}^* y_n = (x^*)^T . y$$

Note that, the dimension of the vectors are $x_{n\times 1}$ and $y_{n\times 1}$, so that it gives a scalar output.

If the $x, y \in \mathbb{R}^n$ then it can stated as $< x, y >= x_1{}^T y_1 + x_2{}^T y_2 + \cdots + x_n{}^T y_n = (x)^T . y$
Moreover, $< x, y >= (x)^T . y =< y, x >= (y)^T . x$.
For $f, g \in \mathcal{C}[a, b]$, the inner product is $< f, g >= \int_a^b f(t)^* g(t) dt$.
Axioms: (i) $(x, x) > 0\ \forall x \neq 0 \iff x = 0$.
(ii) $(x, y) = \overline{(y, x)}\ \forall x, y \in \mathcal{X}$
(iii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)\ \forall x, y, z \in \mathcal{X}; \alpha, \beta \in \mathbb{C}$
(iv) $(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)\ \forall x, y, z \in \mathcal{X}; \alpha, \beta \in \mathbb{C}$
Note: Inner product spaces are also called as Hilbert Spaces. As the inner prıoduct defines norm, then it can be concluded that a Banach Space can be constructed by Hilbert Space. But the reverse is not true as the fact that ''for every norm can not define an inner product''.

Outer Products: $x >< y = x(y^*)^T = \begin{bmatrix} x_1 y_1{}^* & x_1 y_2{}^* & \dots & x_1 y_m{}^* \\ \vdots & \vdots & & \vdots \\ x_n y_1{}^* & x_n y_2{}^* & \dots & x_n y_m{}^* \end{bmatrix}_{n\times m}$

Cross Products: $x \times y = |x||y| sin\theta$

Theorem: (The Parallelogram Law) Let $(\mathcal{X}, <.,.>)$ be inner product space and define

$$\|x\|^2 =< x, x >.$$

Then, for any $x, y \in \mathcal{X}$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Theorem: (The Pythagorean) Let $\mathcal{X}$ be inner product space and for any $x, y \in \mathcal{X}$, if $x \perp y$ then,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Theorem: Let $(\mathcal{X}, \|.\|)$ be a complex normed linear space and for any $x, y \in X$ such that

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Then, $< x, y >= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + j\|x + jy\|^2 - j\|x - jy\|^2)$ defines an inner product on $\mathcal{X}$ such that $\|.\|^2 =<.,.>$. Moreover, the inner product in $(< x, y >)$ is the only one that generates the norm $\|.\|^2$.

Note that, this theorem states the followings:
(i) Given $(\mathcal{X}, <.,.>)$, the norm $\|.\|$ is derived from an inner product (i.e., $\|.\|^2 =< x, x >$) iff the condition $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ is satisfied.
(ii) When $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ is satisfied, the inner product in $< x, y >= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + j\|x + jy\|^2 - j\|x - jy\|^2)$ is the inner product which produces $\|.\|$.

Ex.: Consider $\mathbb{C}^n, \|.\|_2$. Then , it is easy to show that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ is satisfied and when righthand-side of $< x, y >= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + j\|x + jy\|^2 - j\|x - jy\|^2)$ is computed, it gives $\sum_{i=1}^{n} y_i x_i$. Hence 2-norm can be computed from inner product.

Ex.: The inf-norm on $\mathbb{C}^n$ is not derived from an inner product: (Counter ex.) For $n = 2$ assume that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and y= $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So that, $\|x + y\|_\infty^2 + \|x - y\|_\infty^2 = 8$, but $2 * (\|x\|_\infty^2 + \|y\|_\infty^2) = 4$.

Theorem: (Cauchy-Schwarz Inequality) Let $(\mathcal{X}, <.,.>)$ be an inner-product space with $\|x\| = < x, x >^{1/2}$. Then, $|< x, y >| \leq \|x\|\|y\| \; \forall x, y \in \mathcal{X}$. Furthermore, the equality holds iff x and y are linearly dependent.

Proof: Let $t \in \mathbb{R}$ and $x, z \in \mathcal{X}$, $z \neq 0$(otherwise it becomes $0 \leq 0$ which is always true). Then, $p(t) := \|x + tz\|^2 = \|x\|^2 + t < x, z > + t < z, x > + t^2\|z\|^2 = \|x\|^2 + 2 * t * Re < x, z > + t^2\|z\|^2 \geq 0$. Hence, $p(t)$ is quadratic in t and is always nonnegative. This means its discriminant is always non-positive: $(Re < x, z >)^2 \leq \|x\|^2\|y\|^2$.

Ex. : Since $\mathcal{X}$ is a linear vector space and z is arbitrary(not equals to zero), set $z =< x, y > x$ for some $y \in \mathcal{X}$, $y \neq 0$ (otherwise it becomes $0 \leq 0$ which is always true). Noting that,

$$Re \langle x, < x, y > y \rangle = Re(\overline{< x, y >} < x, y >) = \overline{< x, y >} < x, y >= | < x, y > |^2$$

İt follows from the discriminant inequality

$$\begin{aligned} | < x, y > |^4 &\leq \|x\|^2 \langle < x, y > y, < x, y > y \rangle \\ &= \|x\|^2 \; < x, y > \overline{< x, y >} \; \|y\|^2 \\ &= \|x\|^2 \, | < x, y > |^2 \, \|y\|^2 \\ | < x, y > |^2 &\leq \|x\|^2\|y\|^2. \end{aligned}$$

As for the equality version of Cauchy-Schwarz inequality, it occurs iff the discriminant is equal to zero. This condition, combined with $p(t) \geq 0$ implies that $p(t)$ must have a repeated root for some $t_* \in \mathbb{R}$. Therefore, t_* must satisfy $< x + t_* z, x + t_* z \geq 0$, which is true iff $x + t_* z = \mathrm{x} + \mathrm{t} < \mathrm{x}, \mathrm{y} > \mathrm{y} = 0$, which means x and y are linearly dependent.■

Theorem: (Hölder Ine.) Suppose two real numbers p and q ara bigger than one such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, $\int_0^\infty |x(t)y(t)|dt \leq \|x\|_p \|y\|_q$.

Theorem : (Minkowski Ine.) Given p bigger or equal than one such that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Lemma: (Rayleigh-Ritz Inequality) Let $Q = Q^T$and assume that λ_{min} and λ_{max} are minumum and maximum eigenvalues of Q, respectively. Then, the following inequality can be given:

$$\lambda_{min}(Q)\|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|^2$$

Proof: If Q is symmetric then, there exists orthonoal matrix V such that

$$V^T Q V = \Lambda; \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

Multiply this by V and V^T from left and right, respectively :

$$V * V^T * Q * V = V * \Lambda$$
$$Q * V = V * \Lambda$$
$$Q * V * V^T = V * \Lambda * V^T$$
$$Q = V * \Lambda * V^T$$
$$x^T Q x = x^T * V * \Lambda * V^T * x = y^T * \Lambda * y$$

where $x_{n\times1}$, $x^T_{1\times n}$(row vector), $y^T_{1\times n}$(row vector). Hence,

$$[y_1 \quad \cdots \quad y_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2.$$

Take $\lambda_1(\lambda_n)$as maximum eigenvalue(minumum eigenvalue) and we can write that(by 2-norm definition)

$$\lambda_n(y_1^2 + y_2^2 + \cdots + y_n^2) \leq x^T Q x \leq \lambda_1(y_1^2 + y_2^2 + \cdots + y_n^2)$$

$\lambda_{min}(Q)\|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|^2$,(unitary matrices do not change the 2-norm)